

RIGIDITY OF PROPER HOLOMORPHIC MAPS BETWEEN BOUNDED SYMMETRIC DOMAINS

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ABSTRACT. Our first main result gives assumptions guaranteeing that proper holomorphic maps between Cartan type I bounded symmetric domains have simple block matrix shape, answering positively a question of Mok. The proof is based on the second main result establishing similar phenomenon for local CR maps between arbitrary boundary components of two bounded symmetric domains of the above type. Since boundary components other than Shilov boundaries are Levi-degenerate, our analysis is based on their 2-nondegeneracy combining Levi forms with higher order tensors.

1. INTRODUCTION

The goal of this paper is to prove new rigidity results for proper holomorphic maps between bounded symmetric domains. In fact, we obtain our results for maps only defined locally near a boundary point and sending open pieces of boundaries into each other. Furthermore, we also provide a pure CR version of our result for CR maps between boundary components of bounded symmetric domains.

Since the work of Bochner [Bo47] and Calabi [Ca53], rigidity properties of holomorphic isometries between bounded symmetric domains attracted considerable attention. The reader is referred to the survey by Mok [M11] for extensive discussion. See also the work of Siu [S80, S81] for other important rigidity phenomena for bounded symmetric domains, such as the strong rigidity of complex structures of their compact quotients.

Remarkably, many rigidity properties survive when the isometry condition is replaced by purely topological conditions such as *properness*. (Recall that a map between topological spaces is called *proper* if its preimages of compact subsets are compact.) The work on rigidity of *proper holomorphic maps* goes back to the work of Poincaré [P07] and later Alexander [A74] for maps between balls of equal dimension, or more generally, one-sided neighborhoods of their boundary points. However, by intriguing contrast, proper holomorphic maps between balls of *different dimensions* lack similar rigidity properties, see the work of Hakim-Sibony [HS83], Løw [L85], Forstnerič [Fo86a], Globevnik [G87], Stensønes [St96]. On the other hand, rigidity can be regained by strengthening properness by requiring additional boundary regularity, see the work of Webster [W79], Faran [Fa86], Cima-Suffridge [CS83, CS90], Forstnerič [Fo86b, Fo89], Huang [H99, H03], Huang-Ji [HJ01],

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Huang-Ji-Xu [HJX06], Ebenfelt-Minor [EM12] and Ebenfelt [E13]. In another direction, further rigidity phenomena for CR maps between real hypersurfaces and *hyperquadrics* have been discovered by Ebenfelt, Huang and the second author [EHZ04, EHZ05], Baouendi-Huang [BH05], Baouendi-Ebenfelt-Huang [BEH08, BEH09], Ebenfelt-Shroff [ES10] and Ng [Ng13c].

In contrast to holomorphic maps between balls (or CR maps between hypersurfaces), rigidity properties for maps between *bounded symmetric domains* D and D' of *higher rank* are much less understood. If the rank r' of D' does not exceed the rank r of D and both ranks $r, r' \geq 2$, the rigidity of *proper holomorphic maps* $f: D \rightarrow D'$ was conjectured by Mok [M89] and proved by Tsai [Ts93], showing that f is necessarily totally geodesic (with respect to the Bergmann metric). In the remaining case $r < r'$, very little seems to be known, see the work of Tu [Tu02a, Tu02b], Mok [M08] and more recently Mok-Ng-Tu [MNT10], Mok-Ng [MN12], Ng [Ng12, Ng13a, Ng13b].

In [KiZ12], the authors established rigidity for local CR embeddings between Shilov boundaries of Cartan type I bounded symmetric domains $D_{p,q}$ and $D_{p',q'}$ of any rank under the assumption

$$(1.1) \quad p' - q' < 2(p - q)$$

(corresponding to the known assumption $n' < 2n$ for maps between balls in \mathbb{C}^{n+1} and $\mathbb{C}^{n'+1}$ or their boundaries.) Recall that the Cartan type I bounded symmetric domain $D_{p,q}$ is the set of $p \times q$ matrices z over \mathbb{C} such that $I_q - z^*z$ is positive definite, where I_q is the identity $q \times q$ matrix and $z^* = \bar{z}^t$. In [KiZ12], examples were also given of maps of “Whitney type” showing that (1.1) cannot be dropped. However, even though these examples are (polynomial) CR maps between Shilov boundaries, and map $D_{p,q}$ into $D_{p',q'}$, they in general do not induce *proper* maps between these domains, unless the rank $r = q = 1$. Nevertheless, also for proper holomorphic maps between bounded symmetric domains of Cartan type I, rigidity is known to fail (see e.g. [Ts93]) due to the presence of maps of the block matrix form

$$(1.2) \quad f: D_{p,q} \rightarrow D_{p',q'}, \quad z \mapsto \begin{pmatrix} z & 0 \\ 0 & h(z) \end{pmatrix},$$

where $h(z)$ is arbitrary holomorphic matrix-valued function satisfying

$$(1.3) \quad I_{q'-q} - h(z)^*h(z) \text{ is positive definite, } \quad z \in D_{p,q}.$$

In view of this fact, N. Mok asked the following question:

“Are proper holomorphic maps between bounded symmetric domains of higher rank, after composing with suitable automorphisms of the domains, always of the form (1.2)?”

In this paper we consider a situation where we can answer this question affirmatively. In fact, we replace proper maps by more general locally defined ones but have to assume some boundary regularity. Recall (see e.g. [KaZ03]) that the boundary $\partial D_{p,q}$ is a union of q smooth submanifolds (boundary components). We call $x \in \partial D_{p,q}$ a *smooth boundary point*, if $\partial D_{p,q}$ is a smooth hypersurface in a neighborhood of x . As our first main result we prove:

Theorem 1.1. *Let $U \subset \mathbb{C}^{p \times q}$ ($p \geq q > 1$) be an open neighborhood of a smooth boundary point $x \in \partial D_{p,q}$ and $f: U \cap \overline{D_{p,q}} \rightarrow \overline{D_{p',q'}}$ be a smooth map, holomorphic in $U \cap D_{p,q}$ with $f(U \cap \partial D_{p,q}) \subset \partial D_{p',q'}$*

but $f(U \cap D_{p,q}) \not\subset \partial D_{p',q'}$. Assume that

$$(1.4) \quad p' < 2p - 1, \quad q' < p.$$

Then $p' \geq p$, $q' \geq q$ and after composing with suitable automorphisms of $D_{p,q}$ and $D_{p',q'}$, f takes the block matrix form (1.2) with h satisfying (1.3).

Note that the case $q = q' = 1$ corresponds to both $D_{p,q}$ and $D_{p',q'}$ being unit balls, where the same conclusion (also under weaker regularity) is due to Huang [H99]. In this case, the first inequality (1.4) is sharp whereas the second is automatically satisfied. As immediate application of Theorem 1.1 for proper holomorphic maps, we obtain:

Corollary 1.2. *Let $f: D_{p,q} \rightarrow D_{p',q'}$ ($p \geq q > 1$) be a proper holomorphic map which extends smoothly to a neighborhood of a smooth boundary point. Then assuming (1.4) we obtain the conclusion of Theorem 1.1.*

The main difference from the situation of [KiZ12] here is that a proper holomorphic map, even if smoothly extendible to the boundary (and hence sending boundaries into each other), need not send Shilov boundaries into each other, unless the source domain is of rank 1 (i.e. the ball). In higher rank case considered here, boundary extensions of proper holomorphic maps will send boundary components of the source domain into some of those of the target. Thus in order to establish Theorem 1.1, we need to analyze CR maps between general boundary components of $D_{p,q}$ and $D_{p',q'}$. For $p \geq q \geq r \geq 1$, we denote by $S_{p,q,r}$ the *boundary component of rank r* , i.e. the set of all matrices $z \in \partial D_{p,q}$ for which the matrix $I_q - z^*z$ has rank r . We also write $T = TS_{p,q,r}$, $T^c = T^c S_{p,q,r}$, for the tangent and complex tangent spaces and add ' to those for $S_{p',q',r'}$. As our second main result, we prove:

Theorem 1.3. *Let f be a smooth CR map between open pieces of boundary components $S_{p,q,r}$ and $S_{p',q',r'}$ of rank $r < q$ and r' respectively of bounded symmetric domains $D_{p,q}$ and $D_{p',q'}$, such that $df(\xi) \in T' \setminus T'^c$ for any tangent vector $\xi \in T \setminus T^c$. Assume that*

$$(1.5) \quad p' - r' < 2(p - r), \quad q' - r' < p - r.$$

Then $r \leq r'$ and after composing with suitable automorphisms of $D_{p,q}$ and $D_{p',q'}$, f takes the block matrix form

$$(1.6) \quad f(z) = \begin{pmatrix} z & 0 & 0 \\ 0 & I_{r'-r} & 0 \\ 0 & 0 & h(z) \end{pmatrix},$$

where $h: S_{p,q,r} \rightarrow \mathbb{C}^{[(q'-r')-(q-r)] \times [(p'-r')-(p-r)]}$ is a CR map satisfying

$$(1.7) \quad Id - h(z)^* h(z) > 0.$$

Vice versa, for any CR map h satisfying (1.7), f given by (1.6) defines a CR map between open pieces of $S_{p,q,r}$ and $S_{p',q',r'}$.

Comparing to Shilov boundaries $S_{p,q} = S_{p,q,q}$ considered in [KiZ12], the lower rank boundary components $S_{p,q,r}$, $r < q$, present the new substantial difficulty by being *Levi-degenerate*. As a result, similar technique does not lead to desired rigidity. In order to overcome this difficulty, we have to

employ the higher order nondegeneracy (2-nondegeneracy) involving components of different degree, which requires a different approach.

The proofs of Theorems 1.1 and 1.3 are completed in §7.

2. GEOMETRY OF BOUNDARY COMPONENTS

We shall consider the standard inclusion $D_{p,q} \subset \mathbb{C}^{p \times q} \subset Gr(q, p+q)$, where $Gr(q, p+q)$ is the Grassmanian of all q -dimensional subspaces (q -planes) of \mathbb{C}^{p+q} . Here the matrix $z \in \mathbb{C}^{p \times q}$ is identified with the graph in \mathbb{C}^{p+q} of the linear map defined by z . We equip the space \mathbb{C}^{p+q} with the nondegenerate hermitian form

$$(2.1) \quad \langle z, w \rangle = \sum_j \varepsilon_j z_j \bar{w}_j, \quad \varepsilon_j = \begin{cases} -1, & j = 1, \dots, q, \\ 1, & j = q+1, \dots, q+p, \end{cases}$$

called the *basic form*.

In this identification, $D_{p,q}$ is represented by all q -planes $V \subset \mathbb{C}^{p+q}$ such that the restriction $\langle \cdot, \cdot \rangle|_V$ is negative definite, and the boundary component $S_{p,q,r} \subset \partial D_{p,q}$ of rank r by all q -planes $V \subset \mathbb{C}^{p+q}$ such that restriction $\langle \cdot, \cdot \rangle|_V$ has $q-r$ negative and r zero eigenvalues. For $V \in S_{p,q,r}$, denote by $V_0 \subset V$ the r -dimensional kernel of $\langle \cdot, \cdot \rangle|_V$. The connected identity component G of the biholomorphic automorphism group $\text{Aut}(D_{p,q})$ is now identified with the group of all linear transformations of \mathbb{C}^{p+q} preserving $\langle \cdot, \cdot \rangle$, and each $S_{p,q,r}$ is a G -orbit. In this section we will construct a frame bundle over $S_{p,q,r}$ associated with the CR structure of $S_{p,q,r}$ using Grassmannian frames of $Gr(q, p+q)$.

2.1. Adapted frames. An adapted $S_{p,q,r}$ -frame is a set of vectors

$$Z_1, \dots, Z_r, Z'_1, \dots, Z'_{q-r}, X_1, \dots, X_n, Y_1, \dots, Y_r,$$

where

$$n := p - r,$$

for which the basic form is given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & I_r \\ 0 & -I_{q-r} & 0 & 0 \\ 0 & 0 & I_n & 0 \\ I_r & 0 & 0 & 0 \end{pmatrix}.$$

Thus we have

$$V_0 = \text{span}\{Z_1, \dots, Z_r\}, \quad V = V_0 \oplus \text{span}\{Z'_1, \dots, Z'_{q-r}\}$$

and denote

$$V' := \text{span}\{Z'_1, \dots, Z'_{q-r}\}, \quad X := \text{span}\{X_1, \dots, X_n\}, \quad Y := \text{span}\{Y_1, \dots, Y_r\}.$$

The basic form defines the natural duality pairings $V_0 \times Y \rightarrow \mathbb{C}$, $V' \times V' \rightarrow \mathbb{C}$, $X \times X \rightarrow \mathbb{C}$, i.e. we have the identifications

$$(2.2) \quad \overline{V_0} \cong Y^*, \quad \overline{V'} \cong V'^*, \quad \overline{X} \cong X^*,$$

where the “bar” over a complex vector space always denotes the same real vector space with the negative complex structure.

2.2. The tangent space of $S_{p,q,r}$. The tangent space to the Grassmanian $G_{p,q}$ of all q -dimensional subspaces in \mathbb{C}^{p+q} at the element V is isomorphic to $\text{Hom}(V, \mathbb{C}^{p+q}/V)$. Hence, given an adapted frame (Z, Z', X, Y) , it is isomorphic to

$$T_V G_{p,q} = \text{Hom}(V, X \oplus Y).$$

Taking into account the splitting $V = V_0 \oplus V'$, the elements of

$$\text{Hom}(V, X \oplus Y) = \text{Hom}(V_0 \oplus V', X \oplus Y)$$

are given by block 2×2 matrices decomposed as

$$(2.3) \quad R \in \begin{pmatrix} \text{Hom}(V_0, X) & \text{Hom}(V_0, Y) \\ \text{Hom}(V', X) & \text{Hom}(V', Y) \end{pmatrix}.$$

Then the real tangent space $T_V S_{p,q,r}$ to $S_{p,q,r}$ is

$$(2.4) \quad T = T_V S_{p,q,r} = \begin{pmatrix} * & \tilde{R} \\ * & * \end{pmatrix}, \quad \tilde{R} = -\tilde{R}^*,$$

the complex tangent subspace is

$$(2.5) \quad T^c = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}.$$

The complex tangent space T^c contains further two invariantly defined subspaces

$$(2.6) \quad T^- := \{R \in T^c : R(V_0) \subset V\} = \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix}, \quad T^+ := \{R \in T^c : \langle R(V), V_0 \rangle = 0\} = \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix},$$

such that

$$T^+ \cap T^- = T^0, \quad T^+ + T^- = T^c.$$

2.3. The connection matrix form. Write $S := S_{p,q,r}$ and denote by $\mathcal{B} \rightarrow S$ the adapted frame bundle and by π the Maurer-Cartan (connection) form on \mathcal{B} satisfying the symmetry $\pi^* = -\pi$ and the structure equation $d\pi = \pi \wedge \pi$. Then we can write

$$(2.7) \quad \begin{pmatrix} dZ_\alpha \\ dZ'_u \\ dX_k \\ dY_\alpha \end{pmatrix} = \pi \begin{pmatrix} Z_\beta \\ Z'_v \\ X_j \\ Y_\beta \end{pmatrix} = \begin{pmatrix} \psi_\alpha^\beta & \theta_\alpha^v & \theta_\alpha^j & \varphi_\alpha^\beta \\ \sigma_u^\beta & \omega_u^v & \delta_u^j & \theta_u^\beta \\ \sigma_k^\beta & \delta_k^v & \omega_k^j & \theta_k^\beta \\ \xi_\alpha^\beta & \sigma_\alpha^v & \sigma_\alpha^j & \hat{\psi}_\alpha^\beta \end{pmatrix} \begin{pmatrix} Z_\beta \\ Z'_v \\ X_j \\ Y_\beta \end{pmatrix},$$

where the matrix π satisfies the symmetry relation

$$(2.8) \quad \begin{pmatrix} \psi_\alpha^\beta & \theta_\alpha^v & \theta_\alpha^j & \varphi_\alpha^\beta \\ \sigma_u^\beta & \omega_u^v & \delta_u^j & \theta_u^\beta \\ \sigma_k^\beta & \delta_k^v & \omega_k^j & \theta_k^\beta \\ \xi_\alpha^\beta & \sigma_\alpha^v & \sigma_\alpha^j & \hat{\psi}_\alpha^\beta \end{pmatrix} = - \begin{pmatrix} \hat{\psi}_\beta^{\bar{\alpha}} & \epsilon_v \theta_{\bar{v}}^{\bar{\alpha}} & \epsilon_j \theta_{\bar{j}}^{\bar{\alpha}} & \varphi_{\bar{\beta}}^{\bar{\alpha}} \\ \epsilon_u \sigma_{\bar{\beta}}^{\bar{u}} & \epsilon_u \epsilon_v \omega_{\bar{v}}^{\bar{u}} & \epsilon_u \epsilon_j \delta_{\bar{j}}^{\bar{u}} & \epsilon_u \theta_{\bar{\beta}}^{\bar{u}} \\ \epsilon_k \sigma_{\bar{\beta}}^{\bar{k}} & \epsilon_k \epsilon_v \delta_{\bar{v}}^{\bar{k}} & \epsilon_k \epsilon_j \omega_{\bar{j}}^{\bar{k}} & \epsilon_k \theta_{\bar{\beta}}^{\bar{k}} \\ \xi_{\bar{\beta}}^{\bar{\alpha}} & \epsilon_v \sigma_{\bar{v}}^{\bar{\alpha}} & \epsilon_j \sigma_{\bar{j}}^{\bar{\alpha}} & \psi_{\bar{\beta}}^{\bar{\alpha}} \end{pmatrix},$$

where

$$(2.9) \quad \epsilon_u := \langle Z'_u, Z'_u \rangle = -1, \quad u = 1, \dots, q-r, \quad \epsilon_j := \langle X_j, X_j \rangle = 1, \quad j = 1, \dots, n.$$

For instance, differentiating $\langle Z_\alpha, X_j \rangle = 0$ we obtain

$$\langle dZ_\alpha, X_j \rangle + \langle Z_\alpha, dX_j \rangle = 0$$

implying

$$\theta_\alpha^j \langle X_j, X_j \rangle + \theta_j^\alpha \langle Z_\alpha, Y_\alpha \rangle = 0$$

and hence

$$\theta_\alpha^j = -\epsilon_j \theta_j^{\bar{\alpha}}.$$

In the sequel, as in [KiZ12], we shall always work with a local section of the frame bundle $\mathcal{B} \rightarrow S$ and routinely identify forms on \mathcal{B} with their pullbacks to S via that section. With that identification in mind, the forms φ_α^β give a basis in the space of all *contact forms*, i.e. forms vanishing on T^c . Furthermore, the upper right block forms

$$\begin{pmatrix} \theta_\alpha^j & \varphi_\alpha^\beta \\ \delta_u^k & \theta_u^\beta \end{pmatrix}$$

give together a basis in the space of all $(1, 0)$ forms on S .

We shall employ several types of frame changes.

Definition 2.1. We call a change of frame

i) *change of position* if

$$\tilde{Z}_\alpha = W_\alpha^\beta Z_\beta, \quad \tilde{Z}'_u = W_u^\beta Z_\beta + W_u^v Z'_v, \quad \tilde{Y}_\alpha = V_\alpha^\beta Y_\beta + V_\alpha^v Z'_v, \quad \tilde{X}_j = X_j,$$

where $W_0 = (W_\alpha^\beta)$ and $V_0 = (V_\alpha^\beta)$ are $r \times r$ matrices satisfying $V_0^* W_0 = I_r$, $W' = (W_u^v)$ is a $(q-r) \times (q-r)$ matrix satisfying $W'^* W' = I_{q-r}$ and $V_\alpha^\beta W_\beta^*{}^\gamma + V_\alpha^v W_v^*{}^\gamma = 0$;

ii) *change of real vectors* if

$$\tilde{Z}_\alpha = Z_\alpha, \quad \tilde{Z}'_u = Z'_u, \quad \tilde{X}_j = X_j, \quad \tilde{Y}_\alpha = Y_\alpha + H_\alpha^\beta Z_\beta,$$

or

$$(2.10) \quad \begin{pmatrix} \tilde{Z}_\alpha \\ \tilde{Z}'_u \\ \tilde{X}_j \\ \tilde{Y}_\alpha \end{pmatrix} = \begin{pmatrix} I_r & 0 & 0 & 0 \\ 0 & I_{q-r} & 0 & 0 \\ 0 & 0 & I_n & 0 \\ H_\alpha^\beta & 0 & 0 & I_r \end{pmatrix} \begin{pmatrix} Z_\beta \\ Z'_v \\ X_k \\ Y_\beta \end{pmatrix},$$

where $H = (H_\alpha^\beta)$ is a skew hermitian matrix;

iii) *dilation* if

$$\tilde{Z}_\alpha = \lambda_\alpha^{-1} Z_\alpha, \quad \tilde{Z}'_u = Z'_u, \quad \tilde{Y}_\alpha = \lambda_\alpha Y_\alpha, \quad \tilde{X}_j = X_j,$$

where $\lambda_\alpha > 0$;

iv) *rotation* if

$$\tilde{Z}_\alpha = Z_\alpha, \quad \tilde{Z}'_u = Z'_u, \quad \tilde{Y}_\alpha = Y_\alpha, \quad \tilde{X}_j = U_j^k X_k,$$

where (U_j^k) is a unitary matrix.

Consider a change of position as in Definition 2.1. Then φ , θ and δ change to

$$(2.11) \quad \tilde{\varphi}_\alpha^\beta = W_\alpha^\gamma \varphi_\gamma^\delta W_\delta^{*\beta},$$

$$(2.12) \quad \tilde{\theta}_\alpha^j = W_\alpha^\beta \theta_\beta^j,$$

$$(2.13) \quad \tilde{\theta}_u^\alpha = W_u^v \theta_v^\beta W_\beta^{*\alpha} + W_u^\beta \varphi_\beta^\gamma W_\gamma^{*\alpha}$$

$$(2.14) \quad \tilde{\delta}_\alpha^j = W_u^v \delta_v^j,$$

where $W_\delta^{*\beta} = \overline{W_\beta^\delta}$. We shall also make use of the change of frame given by

$$\tilde{Z}_\alpha = Z_\alpha, \quad \tilde{Z}'_u = Z'_u, \quad \tilde{X}_j = X_j + C_j^\beta Z_\beta, \quad \tilde{Y}_\alpha = Y_\alpha + A_\alpha^\beta Z_\beta + B_\alpha^j X_j,$$

or

$$(2.15) \quad \begin{pmatrix} \tilde{Z}_\alpha \\ \tilde{Z}'_u \\ \tilde{X}_j \\ \tilde{Y}_\alpha \end{pmatrix} = \begin{pmatrix} I_r & 0 & 0 & 0 \\ 0 & I_{q-r} & 0 & 0 \\ C_j^\beta & 0 & I_n & 0 \\ A_\alpha^\beta & 0 & B_\alpha^j & I_r \end{pmatrix} \begin{pmatrix} Z_\beta \\ Z'_u \\ X_k \\ Y_\beta \end{pmatrix},$$

such that

$$C_j^\alpha + B_j^\alpha = 0$$

and

$$(A_\alpha^\beta + \overline{A_\beta^\alpha}) + B_\alpha^j B_j^\beta = 0,$$

where

$$B_j^\alpha := \overline{B_\alpha^j}.$$

Then the new frame $(\tilde{Z}, \tilde{Z}', \tilde{Y}, \tilde{X})$ is an $S_{p,q,r}$ -frame. In fact,

$$(2.16) \quad 0 = \langle \tilde{Y}_\alpha, \tilde{Y}_\beta \rangle = \langle Y_\alpha + A_\alpha^\delta Z_\delta + B_\alpha^j X_j, Y_\beta + A_\beta^\gamma Z_\gamma + B_\beta^k X_k \rangle \\ = A_\alpha^\beta \langle Z_\beta, Y_\beta \rangle + \overline{A_\beta^\alpha} \langle Y_\alpha, Z_\alpha \rangle + \sum_j B_\alpha^j \overline{B_\beta^j} \langle X_j, X_j \rangle = (A_\alpha^\beta + \overline{A_\beta^\alpha}) + \sum_j B_\alpha^j \overline{B_\beta^j},$$

and

$$(2.17) \quad 0 = \langle \tilde{X}_j, \tilde{Y}_\alpha \rangle = \langle X_j + C_j^\beta Z_\beta, Y_\alpha + A_\alpha^\delta Z_\delta + B_\alpha^k X_k \rangle = C_j^\alpha \langle Z_\alpha, Y_\alpha \rangle + \overline{B_\alpha^j} \langle X_j, X_j \rangle = C_j^\alpha + \overline{B_\alpha^j},$$

whereas the other scalar products are obviously zero. Furthermore, we claim that the related 1-forms $\tilde{\varphi}_\alpha^\beta$ and $\tilde{\theta}_u^\alpha$ remain the same, while $\tilde{\theta}_\alpha^j$ and $\tilde{\delta}_\alpha^j$ change to

$$(2.18) \quad \tilde{\theta}_\alpha^j = \theta_\alpha^j - \varphi_\alpha^\beta B_\beta^j,$$

$$(2.19) \quad \tilde{\delta}_u^j = \delta_u^j - \theta_u^\beta B_\beta^j.$$

Indeed, differentiation yields

$$(2.20) \quad d\tilde{Z}_\alpha = \tilde{\psi}_\alpha^\beta \tilde{Z}_\beta + \tilde{\theta}_\alpha^v \tilde{Z}'_v + \tilde{\theta}_\alpha^j \tilde{X}_j + \tilde{\varphi}_\alpha^\beta \tilde{Y}_\beta$$

$$(2.21) \quad = \tilde{\psi}_\alpha^\beta Z_\beta + \tilde{\theta}_\alpha^v Z'_v + \tilde{\theta}_\alpha^j (X_j + C_j^\beta Z_\beta) + \tilde{\varphi}_\alpha^\beta (Y_\beta + A_\beta^\gamma Z_\gamma + B_\beta^j X_j)$$

$$(2.22) \quad = dZ_\alpha = \psi_\alpha^\beta Z_\beta + \theta_\alpha^v Z'_v + \theta_\alpha^j X_j + \varphi_\alpha^\beta Y_\beta$$

and

$$(2.23) \quad d\tilde{Z}_u = \tilde{\sigma}_u^\beta \tilde{Z}_\beta + \tilde{\omega}_u^v \tilde{Z}'_v + \tilde{\delta}_u^j \tilde{X}_j + \tilde{\theta}_u^\beta \tilde{Y}_\beta$$

$$(2.24) \quad = \tilde{\sigma}_u^\beta Z_\beta + \tilde{\omega}_u^v Z'_v + \tilde{\delta}_u^j (X_j + C_j^\beta Z_\beta) + \tilde{\theta}_u^\beta (Y_\beta + A_\beta^\gamma Z_\gamma + B_\beta^j X_j)$$

$$(2.25) \quad = dZ_u = \sigma_u^\beta Z_\beta + \omega_u^v Z'_v + \delta_u^j X_j + \theta_u^\beta Y_\beta$$

and the claim follows from identifying the coefficients.

2.4. Structure identities. The structure equations yield

$$(2.26) \quad d\varphi_\alpha^\beta = \theta_\alpha^j \wedge \theta_j^\beta + \theta_\alpha^u \wedge \theta_u^\beta \mod \varphi,$$

$$(2.27) \quad d\theta_\alpha^j = \theta_\alpha^v \wedge \delta_v^j \mod \{\theta_\beta^k, \varphi\},$$

$$(2.28) \quad d\theta_u^\beta = \delta_u^k \wedge \theta_k^\beta \mod \{\theta_v^\alpha, \varphi\},$$

where φ stands for the span of all φ_α^β . The first one via Cartan's formula

$$d\tau(R_1, R_2) = R_1\tau(R_2) - R_2\tau(R_1) - \tau([R_1, R_2]),$$

determines the invariant tensor

$$(2.29) \quad \mathcal{L} = \mathcal{L}_1: T^{1,0} \times T^{1,0} \rightarrow \frac{\mathbb{C}T}{T^{1,0} + T^{0,1}}, \quad (R_1, R_2) \mapsto [R_1, \overline{R}_2] \mod T^{1,0} + T^{0,1},$$

which, in the decomposition (2.4), takes the form

$$(2.30) \quad \left(\begin{pmatrix} a_1 & 0 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ c_2 & d_2 \end{pmatrix} \right) \mapsto a_2^* a_1 - d_1 d_2^* \in \text{Hom}(V_0, Y),$$

and represents the Levi form of S up to imaginary constant. In particular,

$$(2.31) \quad K := \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \subset T^c$$

is the kernel of the Levi form of S . In more invariant terms, \mathcal{L}_1 splits into the sum of two tensors

$$(2.32) \quad \text{Hom}(V_0, X) \times \text{Hom}(V_0, X) \rightarrow \text{Hom}(V_0) \otimes \overline{V}_0, \mathbb{C}, \quad (a_1, a_2) \mapsto \langle a_1, a_2 \rangle,$$

$$(2.33) \quad \text{Hom}(V', Y) \times \text{Hom}(V', Y) \rightarrow \text{Hom}(V_0 \otimes \overline{V}_0, \mathbb{C}), \quad (d_1, d_2) \mapsto -\langle d_2^*, d_1^* \rangle,$$

where we have used the identifications (2.2).

Similarly, (2.27) and (2.28) determine together the invariant tensor

$$(2.34) \quad \mathcal{L}_2: K^{1,0} \times T^{1,0} \rightarrow \frac{T^{1,0}}{K^{1,0}} \cong \frac{T^{1,0} + T^{0,1}}{K^{1,0} + T^{0,1}}, \quad (R_1, R_2) \mapsto [R_1, \overline{R}_2] \mod K^{1,0} + T^{0,1}.$$

Note that since $K^{1,0}$ is in the (complexified) Levi kernel, one always has $[R_1, \overline{R}_2] \subset T^{1,0} + T^{0,1}$. The tensor \mathcal{L}_2 can be regarded as the “second order Levi form” that comes naturally into consideration along with the (first order) Levi form \mathcal{L}_1 to gain the “missing nondegeneracy”. In the decomposition (2.4), \mathcal{L}_2 takes the form

$$(2.35) \quad \left(\begin{pmatrix} 0 & 0 \\ c_1 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ c_2 & d_2 \end{pmatrix} \right) \mapsto (-c_1 d_2^*) \oplus a_2^* c_1 \in \text{Hom}(V_0, X) \oplus \text{Hom}(V', Y),$$

or, in more invariant terms, splits into the sum of two tensors

$$(2.36) \quad \text{Hom}(V', X) \times \text{Hom}(V', Y) \rightarrow \text{Hom}(V_0 \otimes \overline{X}, \mathbb{C}), \quad (c_1, d_2) \mapsto -\langle d_2^*, c_1^* \rangle,$$

$$(2.37) \quad \text{Hom}(V', X) \times \text{Hom}(V_0, X) \rightarrow \text{Hom}(V' \otimes \overline{V}_0, \mathbb{C}), \quad (c_1, a_2) \mapsto \langle c_1, a_2 \rangle,$$

2.5. Important special cases. The case

$$q \leq p, \quad r = 0,$$

corresponds to the Grassmanian of all “maximal negative definite subspaces”, which is the bounded symmetric domain of type $I_{p,q}$, where q is the rank. More generally, the case

$$0 < r \leq q$$

corresponds to the rank r boundary component of the above bounded symmetric domain. Then, in view of (2.32) and (2.33), the tensor \mathcal{L} can be represented by the sesqui-linear map

$$(2.38) \quad \text{Hom}(V_0, V' \oplus X) \times \text{Hom}(V_0, V' \oplus X) \rightarrow \text{Hom}(V_0 \otimes \overline{V}_0, \mathbb{C}), \quad (h_1, h_2) \mapsto \langle h_1, h_2 \rangle_0,$$

where $\langle \cdot, \cdot \rangle_0$ is the standard positive definite hermitian form making the basis Z'_u, X_j orthonormal.

2.6. Structure tensor identities for CR-maps. Let $M = S_{p,q,r}$ and $M' = S_{p',q',r'}$. We shall consider a CR-map $f: M \rightarrow M'$, write latin a, b, c, \dots instead of Greek $\alpha, \beta, \gamma, \dots$, and capital instead of small roman letters for the connection forms on M' and as in [KiZ12], by slight abuse of notation, use the same letters to denote pullbacks of these forms to M via f . The structure equation (2.26) and its analogue for M' imply the equivariance identity for the first structure tensors:

$$(2.39) \quad f_* \mathcal{L}_1(R_1, R_2) = \mathcal{L}'_1(f_* R_1, f_* R_2).$$

As before we identify the complexified normal space $\mathbb{C}T/(T^{1,0} + T^{0,1})$ with $\text{Hom}(V_0 \otimes \overline{V}_0, \mathbb{C})$, i.e. with the space of all sesqui-linear forms on V_0 . Those forms are spanned by the rank one forms $\mu \otimes \overline{\mu}$, where $\mu: V_0 \rightarrow \mathbb{C}$ is a complex-linear functional.

Choose any complex-linear functional $\mu: V' \rightarrow \mathbb{C}$ such that

$$(2.40) \quad f_*(\mu \otimes \overline{\mu}) \neq 0 \in \mathbb{C}TS'/(T^{1,0}S' + T^{0,1}S').$$

Then for the given frame $Z_\alpha, Z'_u, X_j, Y_\alpha$ on S , the rank 1 homomorphisms

$$\mu Z'_u \in \text{Hom}(V', V''), \quad \mu X_j \in \text{Hom}(V', X)$$

yield tangent vectors

$$Z'^\mu_u := \begin{pmatrix} 0 & 0 \\ 0 & (\mu Z'_u)^* \end{pmatrix}, \quad X^\mu_j := \begin{pmatrix} \mu X_j & 0 \\ 0 & 0 \end{pmatrix},$$

which are in view of (2.32) and (2.33), pairwise \mathcal{L} -orthogonal and satisfy

$$\mathcal{L}(Z'_u{}^\mu, Z'_u{}^\mu) = -(\mu \otimes \bar{\mu})\langle Z'_u, Z'_u \rangle = -\epsilon_u(\mu \otimes \bar{\mu}), \quad \mathcal{L}(X_j{}^\mu, X_j{}^\mu) = (\mu \otimes \bar{\mu})\langle X_j, X_j \rangle = \epsilon_j(\mu \otimes \bar{\mu}).$$

In view of (2.39), the push-forwards $f_*Z'_u{}^\mu, f_*X_j{}^\mu$ are pairwise \mathcal{L}' -orthogonal and satisfy

$$(2.41) \quad \mathcal{L}'(f_*Z'_u{}^\mu, f_*Z'_u{}^\mu) = -\epsilon_u f_*(\mu \otimes \bar{\mu}), \quad \mathcal{L}'(f_*X_j{}^\mu, f_*X_j{}^\mu) = \epsilon_j f_*(\mu \otimes \bar{\mu}).$$

3. DETERMINATION OF $\Phi_a{}^b$

3.1. Determination of $\Phi_1{}^1$. Choose a diagonal contact form of M' and say $\Phi_1{}^1$. Since contact forms are spanned by $\varphi_\alpha{}^\beta$, we can write

$$\Phi_1{}^1 = c_\alpha{}^\beta \varphi_\alpha{}^\beta$$

for some smooth functions $c_\alpha{}^\beta$. At generic points, we may assume that either $c_\alpha{}^\beta \equiv 0$ or the matrix $(c_\alpha{}^\beta)$ is of constant rank $l \geq 1$. As in [KiZ12], after a unitary change of frame on M , we obtain

$$\Phi_1{}^1 = \sum_{\alpha=1}^r c_\alpha \varphi_\alpha{}^\alpha$$

for smooth functions c_α . If $c_\alpha{}^\beta \equiv 0$, then $c_\alpha \equiv 0$ for all α and if the matrix $(c_\alpha{}^\beta)$ has constant rank $l \geq 1$, then we may assume that $c_\alpha, \alpha = 1, \dots, l$, never vanish and $c_\alpha \equiv 0$ for $\alpha > l$. Then using (2.26) and its analogue for M' we obtain

$$(3.1) \quad \Theta_1{}^J \wedge \Theta_J{}^1 + \Theta_1{}^U \wedge \Theta_U{}^1 = \sum_{\alpha} c_\alpha (\theta_\alpha{}^J \wedge \theta_J{}^\alpha + \theta_\alpha{}^U \wedge \theta_U{}^\alpha) \mod \varphi,$$

Arguing similar to [KiZ12] we conclude $c_\alpha \geq 0$ and, after dilation, $c_1 = 1$ if $c_1 \neq 0$.

Along with the span φ used before we shall use shortcut notation θ (resp. Θ for M') for the span of the $(1, 0)$ forms $\theta_\alpha{}^j, \theta_u{}^\alpha$. Since in view of (2.41), f sends the Levi kernel of M given by $\varphi = \theta = 0$ into the Levi kernel of M' given by $\Phi = \Theta = 0$, we can write

$$(3.2) \quad \Theta_1{}^J = h_j{}^{J,\alpha} \theta_\alpha{}^j + g_\alpha{}^{J,u} \theta_u{}^\alpha \mod \varphi,$$

$$(3.3) \quad \Theta_U{}^1 = \eta_{U,j}{}^\alpha \theta_\alpha{}^j + \xi_{U,\alpha}{}^u \theta_u{}^\alpha \mod \varphi.$$

Then (3.1) together with symmetry relations (2.8) implies

$$(3.4) \quad \sum_J h_j{}^{J,\alpha} \overline{h_k{}^{J,\beta}} + \sum_U \eta_{U,j}{}^\alpha \overline{\eta_{U,k}{}^\beta} = c_\alpha \hat{\delta}_{\alpha\beta} \hat{\delta}_{jk},$$

$$(3.5) \quad \sum_J h_j{}^{J,\alpha} \overline{g_\beta{}^{J,u}} + \sum_U \eta_{U,j}{}^\alpha \overline{\xi_{U,\beta}{}^u} = 0,$$

$$(3.6) \quad \sum_J g_\alpha{}^{J,u} \overline{g_\beta{}^{J,v}} + \sum_U \xi_{U,\alpha}{}^u \overline{\xi_{U,\beta}{}^v} = c_\alpha \hat{\delta}_{\alpha\beta} \hat{\delta}_{uv},$$

where $\hat{\delta}$ is the Kronecker delta. If $c_\alpha \equiv 0$ for all α , then from (3.4), (3.6) and (3.2), (3.3) we obtain

$$\Theta_1{}^J = \Theta_U{}^1 = 0 \mod \varphi.$$

Now suppose that $c_1 = 1$. Substituting (3.2) and (3.3) respectively into the analogs of (2.27) and (2.28) for M' yields

$$(3.7) \quad h_j^{J,\alpha} d\theta_\alpha^j + g_\alpha^{J,u} d\theta_u^\alpha = (\eta_\alpha^{V,j} \theta_j^\alpha + \xi_u^{V,\alpha} \theta_\alpha^u) \wedge \Delta_V^J \mod \theta, \varphi,$$

$$(3.8) \quad \eta_{U,j}^\alpha d\theta_\alpha^j + \xi_{U,\alpha}^u d\theta_u^\alpha = \Delta_U^J \wedge (h_{J,\alpha}^j \theta_j^\alpha + g_{J,u}^\alpha \theta_\alpha^u) \mod \theta, \varphi,$$

where

$$(3.9) \quad \eta_\alpha^{U,j} := -\overline{\eta_{U,j}^\alpha}, \quad \xi_u^{U,\alpha} := \overline{\xi_{U,\alpha}^u}, \quad h_{J,\alpha}^j := \overline{h_j^{J,\alpha}}, \quad g_{J,u}^\alpha := -\overline{g_\alpha^{J,u}}.$$

Using (2.27) and (2.28), we rewrite (3.7) and (3.8) as

$$(3.10) \quad h_j^{J,\alpha} \theta_\alpha^u \wedge \delta_u^j + g_\alpha^{J,u} \delta_u^j \wedge \theta_j^\alpha = (\eta_\alpha^{V,j} \theta_j^\alpha + \xi_u^{V,\alpha} \theta_\alpha^u) \wedge \Delta_V^J \mod \theta, \varphi,$$

$$(3.11) \quad \eta_{U,j}^\alpha \theta_\alpha^u \wedge \delta_u^j + \xi_{U,\alpha}^u \delta_u^j \wedge \theta_j^\alpha = \Delta_U^J \wedge (h_{J,\alpha}^j \theta_j^\alpha + g_{J,u}^\alpha \theta_\alpha^u) \mod \theta, \varphi.$$

By Cartan's Lemma,

$$(3.12) \quad \eta_\alpha^{V,j} \Delta_V^J = -g_\alpha^{J,u} \delta_u^j \mod \theta, \bar{\theta}, \varphi,$$

$$(3.13) \quad g_{J,u}^\alpha \Delta_U^J = -\eta_{U,j}^\alpha \delta_u^j \mod \theta, \bar{\theta}, \varphi,$$

For α, j fixed, consider vector $\eta_\alpha^j := (\eta_\alpha^{U,j})_U \in \mathbb{C}^{q'-r'}$. Since $q' - r' < n$ by (1.5), these vectors are linearly dependent, i.e. $\sum d_j \eta_\alpha^j = 0$ for some $(d_1, \dots, d_n) \neq 0$. Then by (3.12),

$$(3.14) \quad g_\alpha^{J,u} d_j \delta_u^j = 0 \mod \theta, \bar{\theta}, \varphi.$$

Since $q - r \geq 1$ and $\delta_u^j, 1 \leq u \leq q - r, 1 \leq j \leq n$, are linearly independent modulo $\theta, \bar{\theta}, \varphi$, it follows that $g_\alpha^{J,u} = 0$, and hence $\eta_{U,j}^\alpha = 0$ by (3.13).

Now it follows from (3.4) that the vectors $h_j^\alpha := (h_j^{J,\alpha})_J$ are pairwise orthogonal and have length c_α independent of j . Then after a unitary rotation of the frame as in [KiZ12] (proof of Lemma 4.1), we may assume that the vectors h_j^1 of length $c_1 = 1$ are precisely the first n standard vectors in $\mathbb{C}^{n'}$. Using $n' < 2n$ and repeating the rest of the argument in [KiZ12], we conclude $s = 1$ and hence

$$(3.15) \quad \Phi_1^1 = \varphi_1^1,$$

$$(3.16) \quad \Theta_1^J = \theta_1^J \mod \varphi.$$

Finally (3.6) is now of the form

$$\sum_U \xi_{U,\alpha}^u \overline{\xi_{U,\beta}^v} = \hat{\delta}_{1\alpha} \hat{\delta}_{\alpha\beta} \hat{\delta}_{uv}.$$

Then after a unitary change of frame we obtain

$$(3.17) \quad \Theta_U^1 = \theta_U^1 \mod \varphi.$$

Summarizing we obtain

Lemma 3.1.

$$(3.18) \quad \Theta_1^J = c_1 \theta_1^J \mod \varphi,$$

$$(3.19) \quad \Theta_U^1 = c_1 \theta_U^1 \mod \varphi,$$

where $\Phi_1^1 = c_1 \varphi_1^1$, and c_1 is either 0 or 1.

Furthermore, by considering Φ_a^a for arbitrary a , we can show the following lemma.

Lemma 3.2. *Let θ^+ and θ^- be ideals generated by θ_α^j and θ_u^β respectively. Then*

$$(3.20) \quad \Theta_a^j = 0 \pmod{\theta^+, \varphi},$$

$$(3.21) \quad \Theta_U^a = 0 \pmod{\theta^-, \varphi},$$

i.e., the subspace T^+ and T^- are preserved by f . That is,

$$f_*(T^+) \subset T'^+, \quad f_*(T^-) \subset T'^-.$$

3.2. Determination of Φ_2^2 and Φ_2^1 . Suppose first $\Phi_a^a \equiv 0$ for all a . Then Lemma 3.1 and its analogues for $\alpha = 2, \dots, r$, we obtain

$$(3.22) \quad \Phi \equiv 0$$

$$(3.23) \quad \Theta \equiv 0 \pmod{\varphi}.$$

Now assume that there exists a such that $\Phi_a^a \neq 0$, say $a = 1$. Then

$$\Phi_1^1 = \varphi_1^1$$

by Lemma 3.1 and let

$$(3.24) \quad \Phi_a^1 = \lambda_a \varphi_1^1 \pmod{\{\varphi_\alpha^\beta, \alpha \geq 2 \text{ or } \beta \geq 2\}}, \quad a \geq 2,$$

for some smooth functions λ_a , $a = 2, \dots, r'$. Then (2.26) and its analogue for M' together with Lemmas 3.1 and 3.2 imply

$$(3.25) \quad \Theta_a^j \wedge \theta_j^1 = \lambda_a \theta_1^j \wedge \theta_j^1 \pmod{\theta_\alpha, \overline{\theta}_\alpha, \alpha \geq 2, \theta^-, \overline{\theta}^-, \varphi, \quad a \geq 2},$$

where θ_α is the span of all θ_α^j .

Then there exists a change of position (see Definition 2.1) that leaves Θ_1^j invariant and replaces Θ_a^j with $\Theta_a^j - \lambda_a \Theta_1^j$, $a \geq 2$, (see the discussion after Definition 2.1). The same change of position leaves Φ_1^1 invariant and transforms Φ_a^1 into $\Phi_a^1 - \lambda_a \Phi_1^1$ for $a \geq 2$. After performing such change of position, (3.24) becomes

$$\Phi_a^1 = 0 \pmod{\{\varphi_\alpha^\beta : \alpha \geq 2 \text{ or } \beta \geq 2\}}, \quad a \geq 2,$$

and (3.25) becomes

$$(3.26) \quad \Theta_a^j \wedge \theta_j^1 + \Theta_a^u \wedge \theta_u^1 = 0 \pmod{\theta_\alpha, \overline{\theta}_\alpha, \alpha \geq 2, \theta^-, \overline{\theta}^-, \varphi, \quad a \geq 2}.$$

Since Θ_a^j , θ_u^1 are $(1, 0)$ but Θ_a^u , θ_j^1 are $(0, 1)$ and linearly independent, it follows from Cartan's lemma together with Lemma 3.2 that

$$(3.27) \quad \Theta_a^j = \Theta_u^a = 0 \pmod{\{\theta_\alpha, \theta_v^\alpha : \alpha \geq 2\}, \varphi, \quad a \geq 2}.$$

Since Θ_a^j are spanned by θ^+ and Θ_u^a are spanned by θ^- , we conclude that

$$(3.28) \quad \Theta_a^j = 0 \pmod{\{\theta_\alpha, : \alpha \geq 2\}, \varphi, \quad a \geq 2}$$

$$(3.29) \quad \Theta_u^a = 0 \pmod{\{\theta_v^\alpha : \alpha \geq 2\}, \varphi, \quad a \geq 2}.$$

Next for each $a \geq 2$, let

$$(3.30) \quad \Phi_a^a = \lambda_{a,\beta} \varphi_1^\beta \pmod{\{\varphi_\alpha^\gamma : \alpha \geq 2\}}$$

for some functions $\lambda_{a,\beta}$. Suppose first that there exists a and β such that $\lambda_{a,\beta} \neq 0$. We may assume $a = 2$. Using the identity

$$d\Phi_2^2 = \Theta_2^J \wedge \Theta_J^2 \mod \Phi, \Theta^-,$$

together with (3.28) and (2.26) we obtain

$$(3.31) \quad \sum_{J=n+1}^{n'} \Theta_2^J \wedge \Theta_J^2 = \lambda_{2,\beta} \theta_1^j \wedge \theta_j^\beta \mod \{\theta_\alpha, \theta^- : \alpha \geq 2\}, \varphi,$$

where $\lambda_{2,\beta} \neq 0$ for some fixed β . On the left-hand side we have a linear combination of $n' - n$ $(1, 0)$ forms, whereas on the right-hand side we have a linear combination of at least n linear independent $(1, 0)$ forms with nonzero coefficients. Since $n' - n < n$, Cartan's lemma implies that this is impossible. Hence we have $\lambda_{a,\beta} = 0$ for all $a \geq 2$ and all β and therefore (3.30) implies

$$\Phi_a^a = 0 \mod \{\varphi_\alpha^\beta : \alpha \geq 2\}, \quad a \geq 2.$$

Since Φ_a^b and φ_α^β are antihermitian, we also have

$$\Phi_a^a = 0 \mod \{\varphi_\alpha^\beta : \beta \geq 2\}, \quad a \geq 2,$$

and hence

$$(3.32) \quad \Phi_a^a = 0 \mod \{\varphi_\alpha^\beta : \alpha, \beta \geq 2\}, \quad a \geq 2.$$

Now (2.26), (3.28), (3.29) and Lemma 3.2 imply

$$(3.33) \quad \sum_{J=n+1}^{n'} \Theta_a^J \wedge \overline{\Theta_a^J} + \sum_{U=q-r+1}^{q'-r'} \overline{\Theta_a^U} \wedge \Theta_a^U = 0 \mod \{\theta_\alpha, \theta_u^\alpha : \alpha \geq 2\}, \varphi, \quad a \geq 2,$$

which in view of Lemma 3.2 and positivity of the left-hand side implies

$$(3.34) \quad \Theta_a^J = 0 \mod \{\theta_\alpha : \alpha \geq 2\}, \varphi, \quad a \geq 2, J > n,$$

$$(3.35) \quad \Theta_U^a = 0 \mod \{\theta_u^\alpha : \alpha \geq 2\}, \varphi, \quad a \geq 2, U > q - r.$$

Together with (3.28) and (3.29) this yields

$$(3.36) \quad \Theta_a^J = 0 \mod \{\theta_\alpha : \alpha \geq 2\}, \varphi, \quad a \geq 2,$$

$$(3.37) \quad \Theta_U^a = 0 \mod \{\theta_u^\alpha : \alpha \geq 2\}, \varphi, \quad a \geq 2.$$

Now we redo our procedure for Φ_a^b . We can write

$$(3.38) \quad \Phi_a^b = \lambda_a^{b\alpha} \varphi_\alpha^\beta$$

for which (2.26) yields

$$(3.39) \quad \Theta_a^J \wedge \Theta_J^b + \Theta_a^U \wedge \Theta_U^b = \lambda_a^{b\alpha} (\theta_\alpha^j \wedge \theta_j^\beta + \theta_\alpha^u \wedge \theta_u^\beta) \mod \varphi.$$

Then substituting (3.36) we obtain

$$(3.40) \quad \lambda_a^{b\alpha} \theta_\alpha^j \wedge \theta_j^\beta = 0 \mod \{\theta_\gamma^k \wedge \theta_l^\delta, \theta_\gamma^u \wedge \theta_v^\delta : \gamma, \delta \geq 2\}, \varphi, \quad a, b \geq 2,$$

which implies

$$(3.41) \quad \lambda_a^{b\alpha} = \lambda_a^{b1} = 0, \quad a, b \geq 2.$$

Hence (3.38) yields

$$(3.42) \quad \Phi_a^b = 0 \pmod{\{\varphi_\alpha^\beta : \alpha, \beta \geq 2\}}, \quad a, b \geq 2.$$

Summarizing we obtain the following:

$$(3.43) \quad \Phi_a^1 = 0 \pmod{\{\varphi_\alpha^\beta : \alpha \geq 2 \text{ or } \beta \geq 2\}}, \quad a \geq 2,$$

$$(3.44) \quad \Phi_a^b = 0 \pmod{\{\varphi_\alpha^\beta : \alpha, \beta \geq 2\}}, \quad a, b \geq 2,$$

$$(3.45) \quad \Theta_a^J = 0 \pmod{\{\theta_\alpha : \alpha \geq 2\}, \varphi}, \quad a \geq 2,$$

$$(3.46) \quad \Theta_U^a = 0 \pmod{\{\theta_u^\alpha : \alpha \geq 2, u > r\}, \varphi}.$$

Now repeat the argument from the beginning of this section and assume first that $\Phi_a^a = 0$ for all $a \geq 2$. We obtain

$$\Theta_a^J = \Theta_U^a = 0 \pmod{\varphi}, \quad a \geq 2,$$

and hence $d\Phi_a^b$ vanishes on the kernel of all θ_1^j , θ_u^1 and φ_α^β . In this case (3.38) and (3.39) imply

$$\Phi_a^b = 0, \quad a > 1 \quad \text{or} \quad b > 1.$$

In the remaining case, we assume that $\Phi_a^a \neq 0$ for some a , say $a = 2$. Then (3.32) implies that, after a change of position as before, we may assume that

$$\Phi_2^2 = \sum_{\alpha \geq 2} c_\alpha \varphi_\alpha^\alpha$$

for some $c_\alpha \geq 0$ not all zero. Then (2.26) yields

$$(3.47) \quad \Theta_2^J \wedge \Theta_J^2 + \Theta_2^U \wedge \Theta_U^2 = \sum_{\alpha \geq 2} c_\alpha (\theta_\alpha^j \wedge \theta_j^\alpha + \theta_\alpha^u \wedge \theta_u^\alpha) \pmod{\varphi}.$$

Since the proof of Lemma 3.1 can be repeated for Φ_2^2 instead of Φ_1^1 , we conclude that the rank of the left-hand side of (3.47) restricted to T^1 is n . Therefore, in the right-hand side, only one c_α , say c_2 can be different from zero. After a dilation (see Definition 2.1), we may assume

$$\Phi_2^2 = \varphi_2^2$$

and hence

$$(3.48) \quad \sum_J \Theta_2^J \wedge \overline{\Theta_2^J} + \sum_U \overline{\Theta_U^2} \wedge \Theta_U^2 = \sum_j \theta_2^j \wedge \overline{\theta_2^j} + \sum_u \overline{\theta_u^2} \wedge \theta_u^2 \pmod{\varphi}.$$

We claim that each Θ_2^J and Θ_U^2 is a linear combination of only θ_2^j and θ_u^2 modulo φ . Indeed, if Θ_2^J were a combination of θ_α^j modulo φ , where some of them enters with a nonzero coefficient λ_α with $\alpha \neq 2$, we would have $\theta_\alpha^j \wedge \overline{\theta_\alpha^j}$ entering with positive coefficient $\geq \lambda_\alpha \overline{\lambda_\alpha}$ in the right-hand side of (3.48), which is impossible. Similar argument for Θ_U^2 proves our claim. As in the proof of Lemma 3.1 we now write

$$(3.49) \quad \Theta_2^J = h^J \theta_2^j \pmod{\varphi}.$$

Since

$$(3.50) \quad \Phi_2^1 = \lambda_\beta^\alpha \varphi_\alpha^\beta$$

for suitable λ_α^β , we obtain

$$(3.51) \quad \Theta_2^J \wedge \Theta_J^1 = \lambda_\beta^\alpha \theta_\alpha^j \wedge \theta_j^\beta \mod \theta_\gamma^u \wedge \theta_v^\delta, \varphi,$$

which in view of (3.49) and Lemma 3.1, yields

$$(3.52) \quad h_j^k \theta_2^j \wedge \theta_k^1 = \lambda_\beta^\alpha \theta_\alpha^j \wedge \theta_j^\beta \mod \theta_\gamma^u \wedge \theta_v^\delta, \varphi.$$

Since the right-hand side contains no terms $\theta_\alpha^j \wedge \theta_k^\beta$ with $j \neq k$, it follows that $h_j^k = 0$ for $j \neq k$ and hence $h_j^j = \lambda_2^1 =: \lambda$ for all j and $\lambda_\beta^\alpha = 0$ for $(\alpha, \beta) \neq (1, 2)$. Then (3.49) implies

$$(3.53) \quad \Theta_2^j = \lambda \theta_2^j \mod \varphi.$$

Finally, substituting (3.53) into (3.48) and identifying coefficients we obtain

$$\lambda \bar{\lambda} \delta_{ij} + \sum_{J>n} h_i^J \overline{h_j^J} = \delta_{ij}.$$

In particular, it follows that the vectors $h_i := (h_i^{n+1}, \dots, h_i^{n'})$ are orthogonal and of the same length. But since we have assumed $n' - n < n$, we must have $h_i = 0$ and therefore $|\lambda| = 1$. Now we perform a change of position as in Definition 2.1 with $W_\alpha^\beta := c_\alpha \delta_{\alpha\beta}$ with $c_\alpha = 1$ for $\alpha \neq 2$ and $c_2 = \lambda$. Then we arrive at the following relations:

$$(3.54) \quad \Phi_a^a = \varphi_a^a, \quad a = 1, 2,$$

$$(3.55) \quad \Theta_a^J = \theta_a^J \mod \varphi, \quad a = 1, 2.$$

Since after the last change of position, we have $\lambda = 1$ in (3.53), we obtain from (3.52) that $\lambda_\beta^\alpha = 0$ unless $\alpha = 2$ and $\beta = 1$, in which case $\lambda_2^1 = 1$. Then substituting into (3.50) yields

$$(3.56) \quad \Phi_2^1 = \varphi_2^1.$$

Finally (2.26), (3.6) and Lemma 3.1 imply

$$(3.57) \quad \Theta_U^a = \theta_U^a \mod \varphi, \quad a = 1, 2.$$

3.3. Determination of Φ_a^b . Now we repeat again the arguments after the proof of Lemma 3.1, where we replace 1 by 2 and 2 by 3, to arrive at the identities:

$$(3.58) \quad \Phi_a^2 = 0 \mod \{\varphi_\alpha^\beta : \alpha \geq 3 \text{ or } \beta \geq 3\}, \quad a \geq 3,$$

$$(3.59) \quad \Phi_a^b = 0 \mod \{\varphi_\alpha^\beta : \alpha, \beta \geq 3\}, \quad a, b \geq 3,$$

$$(3.60) \quad \Theta_a^J = 0 \mod \{\theta_\alpha : \alpha \geq 3\}, \varphi, \quad a \geq 3,$$

$$(3.61) \quad \Theta_U^a = 0 \mod \{\theta_u^\alpha : \alpha \geq 3, u > r\}, \varphi, \quad a \geq 3.$$

Then continuing following the arguments after (3.46) with the same replacements, we obtain

$$(3.62) \quad \Phi_3^\alpha = \varphi_3^\alpha, \quad \alpha = 1, 2, 3,$$

$$(3.63) \quad \Theta_3^J - \theta_3^J = \Theta_U^3 - \theta_U^3 = 0 \mod \varphi.$$

Finally, arguing by induction on $b = 4, \dots, q'$, and proceeding by repeating the same arguments with 1 replaced by b and 2 by $b+1$, we obtain the following lemma.

Lemma 3.3. *For any local CR mapping f from $S_{p,q,r}$ into $S_{p',q',r'}$, there exist an integer $s \leq \min(r, r')$ and a choice of sections of the bundles $\mathcal{B}_{p,q,r} \rightarrow S_{p,q,r}$ and $\mathcal{B}_{p',q',r'} \rightarrow S_{p',q',r'}$ such that the pulled back forms satisfy*

$$\begin{aligned}\Phi_a^b - \tilde{\varphi}_a^b &= 0, \\ \Theta_a^J - \tilde{\theta}_a^J &= \Theta_U^a - \tilde{\theta}_U^a = 0 \pmod{\varphi},\end{aligned}$$

where

$$(3.64) \quad \tilde{\varphi}_a^b := \varphi_a^b \quad \text{if } a, b \leq s,$$

$$(3.65) \quad \tilde{\theta}_a^J := \theta_a^J \quad \text{if } a \leq s, J \leq n,$$

$$(3.66) \quad \tilde{\theta}_U^a := \theta_U^a \quad \text{if } a \leq s, U \leq q - r,$$

and 0 otherwise.

4. DETERMINATION OF Θ .

Our next goal is to determine Θ_a^J and Θ_U^a . It will be determined together with components Ψ , Δ and Ω modulo φ . In view of Lemma 3.3 we can write

$$(4.1) \quad \Theta_a^J - \tilde{\theta}_a^J = \eta_a^J{}_\beta{}^\gamma \varphi_\gamma^\beta,$$

$$(4.2) \quad \Theta_U^a - \tilde{\theta}_U^a = \eta_U^a{}_\beta{}^\gamma \varphi_\gamma^\beta,$$

for some $\eta_a^J{}_\beta{}^\gamma, \eta_U^a{}_\beta{}^\gamma$.

4.1. Determination of Θ_a^J, Θ_U^a for $a > s$. In case $a > s$, differentiating (4.1), (4.2) and using the structure equations $d\pi = \pi \wedge \pi$ for M' , we obtain

$$(4.3) \quad \eta_a^J{}_\beta{}^\gamma \left(\theta_\gamma^v \wedge \theta_v^\beta + \theta_\gamma^k \wedge \theta_k^\beta \right) = \Psi_a^b \wedge \tilde{\theta}_b^J \pmod{\varphi},$$

$$(4.4) \quad \eta_U^a{}_\beta{}^\gamma \left(\theta_\gamma^v \wedge \theta_v^\beta + \theta_\gamma^k \wedge \theta_k^\beta \right) = \tilde{\theta}_U^b \wedge \hat{\Psi}_b^a \pmod{\varphi}.$$

If $J > n$ and $U > q - r$, the right-hand sides of (4.3) and (4.4) are zero. Since the forms $\theta_\gamma^k, \theta_u^\beta$ and $\theta_k^\beta, \theta_\gamma^u$ are $(1, 0)$ and $(0, 1)$ respectively, and are linearly independent, we conclude

$$\eta_a^J{}_\beta{}^\gamma = \eta_U^a{}_\beta{}^\gamma = 0, \quad a > s, J > n, U > q - r,$$

and hence (4.1) and (4.2) yield

$$(4.5) \quad \Theta_a^J = \Theta_U^a = 0, \quad a > s, J > n, U > q - r.$$

For $J = j \leq n$ and $U = u \leq q - r$, (4.3) and (4.4) take the form

$$(4.6) \quad \eta_a^j{}_\beta{}^\gamma \left(\theta_\gamma^v \wedge \theta_v^\beta + \theta_\gamma^k \wedge \theta_k^\beta \right) = \sum_{b \leq s} \Psi_a^b \wedge \theta_b^j \pmod{\varphi},$$

$$(4.7) \quad \eta_u^a{}_\beta{}^\gamma \left(\theta_\gamma^v \wedge \theta_v^\beta + \theta_\gamma^k \wedge \theta_k^\beta \right) = \sum_{b \leq s} \theta_u^b \wedge \hat{\Psi}_b^a \pmod{\varphi}.$$

Since the forms θ_γ^k , θ_v^β both appear on the left-hand side of (4.6) while only θ_β^j appears on the right-hand side of (4.6), we obtain

$$\eta_a^j{}_\beta{}^\gamma = 0.$$

Similar argument for (4.7) yields

$$\eta_u^a{}_\beta{}^\gamma = 0.$$

Hence by (4.1) and (4.2), we obtain

$$(4.8) \quad \Theta_a^j = \Theta_u^a = 0, \quad a > s.$$

Furthermore by substituting $\eta_a^j{}_\beta{}^\gamma = \eta_u^a{}_\beta{}^\gamma = 0$, (4.6) and (4.7) yield

$$\Psi_a^b = 0 \mod \{\theta_\beta^j, \varphi\}, \quad \text{if } a > s \text{ and } b \leq s$$

and

$$\hat{\Psi}_b^a = 0 \mod \{\theta_u^\beta, \varphi\}, \quad \text{if } a > s \text{ and } b \leq s.$$

Then by symmetry relation for Ψ and $\hat{\Psi}$, we obtain

$$(4.9) \quad \Psi_a^b = 0 \mod \varphi, \quad \text{if } a > s \text{ and } b \leq s.$$

4.2. Reducing the freedom for Θ_a^J , Θ_U^a for $a \leq s$. In case $a = \alpha \leq s$, differentiating (4.1), (4.2) and using the structure equations $d\pi = \pi \wedge \pi$ for both M and M' , we obtain

$$(4.10) \quad \begin{aligned} & \eta_\alpha^J{}_\beta{}^\gamma \left(\theta_\gamma^v \wedge \theta_v^\beta + \theta_\gamma^k \wedge \theta_k^\beta \right) + \sum_{\beta > s} \psi_\alpha^\beta \wedge \theta_\beta^J \\ &= \sum_{\beta \leq s} (\Psi_\alpha^\beta - \psi_\alpha^\beta) \wedge \theta_\beta^J + \theta_\alpha^v \wedge (\Delta_v^J - \delta_v^J) + \theta_\alpha^k \wedge (\Omega_k^J - \omega_k^J) \mod \varphi, \end{aligned}$$

$$(4.11) \quad \begin{aligned} & \eta_U^{\alpha}{}_\beta{}^\gamma \left(\theta_\gamma^v \wedge \theta_v^\beta + \theta_\gamma^k \wedge \theta_k^\beta \right) + \sum_{\beta > s} \theta_U^\beta \wedge \hat{\psi}_\beta^\alpha \\ &= (\Omega_U^v - \omega_U^v) \wedge \theta_v^\alpha + (\Delta_U^k - \delta_U^k) \wedge \theta_k^\alpha + \sum_{\beta \leq s} \theta_U^\beta \wedge \left(\hat{\Psi}_\beta^\alpha - \hat{\psi}_\beta^\alpha \right) \mod \varphi. \end{aligned}$$

Since the forms θ_γ^u and θ_β^k are $(0, 1)$ and $(1, 0)$ respectively and are linearly independent, the terms $\theta_\gamma^v \wedge \theta_v^\beta$, $\gamma \neq \alpha$, in the left-hand of (4.10) side cannot occur in the right-hand side. Therefore

$$(4.12) \quad \eta_\alpha^J{}_\beta{}^\gamma = 0 \quad \text{if } \gamma \neq \alpha$$

and hence (4.1) becomes

$$(4.13) \quad \Theta_\alpha^J - \tilde{\theta}_\alpha^J = \eta_\alpha^J{}_\beta{}^\alpha \varphi_\beta^\alpha, \quad \alpha \leq s,$$

where

$$\eta_\alpha^J{}_\beta{}^\alpha := \eta_\alpha^J{}_\beta{}^\alpha.$$

Similar argument for $\theta_\gamma^k \wedge \theta_k^\beta$ in (4.11) yields

$$\eta_U^{\alpha}{}_\beta{}^\gamma = 0 \quad \text{if } \beta \neq \alpha$$

and hence (4.2) becomes

$$(4.14) \quad \Theta_U^\alpha - \tilde{\theta}_U^\alpha = \eta_U^{\alpha\gamma} \varphi_\gamma^\alpha, \quad \alpha \leq s,$$

where

$$\eta_U^{\alpha\gamma} := \eta_U^{\alpha\gamma}.$$

Now if $J > n$ and $U > q - r$, then (4.10) and (4.11) become

$$(4.15) \quad \eta_{\alpha\beta}^J \left(\theta_\alpha^v \wedge \theta_v^\beta + \theta_\alpha^k \wedge \theta_k^\beta \right) = \theta_\alpha^v \wedge \Delta_v^J + \theta_\alpha^k \wedge \Omega_k^J \mod \varphi,$$

i.e.

$$(4.16) \quad \theta_\alpha^v \wedge (\Delta_v^J - \eta_{\alpha\beta}^J \theta_v^\beta) + \theta_\alpha^k \wedge (\Omega_k^J - \eta_{\alpha\beta}^J \theta_k^\beta) = 0 \mod \varphi,$$

and

$$(4.17) \quad \eta_U^{\alpha\gamma} (\theta_\gamma^v \wedge \theta_v^\alpha + \theta_\gamma^k \wedge \theta_k^\alpha) = \Omega_U^v \wedge \theta_v^\alpha + \Delta_U^k \wedge \theta_k^\alpha \mod \varphi,$$

i.e.

$$(4.18) \quad (\Omega_U^v - \eta_U^{\alpha\gamma} \theta_\gamma^v) \wedge \theta_v^\alpha + (\Delta_U^k - \eta_U^{\alpha\gamma} \theta_\gamma^k) \wedge \theta_k^\alpha = 0 \mod \varphi.$$

Thus using linear independence of θ_α^k , θ_v^α and applying Cartan's Lemma, we obtain for each α the identities

$$(4.19) \quad \Delta_v^J = \eta_{\alpha\beta}^J \theta_v^\beta \mod \{\varphi, \theta_\alpha^+, \theta_\alpha^-\}, \quad J > n,$$

$$(4.20) \quad \Omega_k^J = \eta_{\alpha\beta}^J \theta_k^\beta \mod \{\varphi, \theta_\alpha^+, \theta_\alpha^-\}, \quad J > n,$$

and

$$(4.21) \quad \Delta_U^k = \eta_U^{\alpha\gamma} \theta_\gamma^k \mod \{\varphi, \overline{\theta_\alpha^+}, \overline{\theta_\alpha^-}\}, \quad U > q - r,$$

$$(4.22) \quad \Omega_U^v = \eta_U^{\alpha\gamma} \theta_\gamma^v \mod \{\varphi, \overline{\theta_\alpha^+}, \overline{\theta_\alpha^-}\}, \quad U > q - r,$$

where θ_α^+ , θ_α^- are spans of θ_α^k and θ_α^v respectively. Since Δ_v^J , Δ_U^k are $(1, 0)$ forms, θ_α^v , θ_k^α are $(0, 1)$, and Δ_v^J , Δ_U^k and Ω_k^J , Ω_U^v are independent of α , we obtain

$$(4.23) \quad \Delta_v^J = \eta_\beta^J \theta_v^\beta \mod \{\varphi, \theta_\alpha^+\}, \quad J > n,$$

$$(4.24) \quad \Omega_k^J = \eta_\beta^J \theta_k^\beta \mod \{\varphi, \theta_\alpha^+, \theta_\alpha^-\}, \quad J > n,$$

$$(4.25) \quad \Delta_U^k = \eta_U^\gamma \theta_\gamma^k \mod \{\varphi, \overline{\theta_\alpha^-}\}, \quad U > q - r,$$

$$(4.26) \quad \Omega_U^v = \eta_U^\gamma \theta_\gamma^v \mod \{\varphi, \overline{\theta_\alpha^+}, \overline{\theta_\alpha^-}\}, \quad U > q - r.$$

where

$$\eta_\beta^J = \eta_{\alpha\beta}^J, \quad \eta_U^\gamma = \eta_U^{\alpha\gamma}$$

and (4.13) and (4.14) become

$$(4.27) \quad \Theta_\alpha^J = \eta_\beta^J \varphi_\alpha^\beta, \quad J > n,$$

$$(4.28) \quad \Theta_U^\alpha = \eta_U^\gamma \varphi_\gamma^\alpha, \quad U > q - r.$$

If on the other hand, $J = j \leq n$ and $U = u \leq q - r$, then (4.10) together with (4.12) yields

$$(4.29) \quad \theta_\alpha^v \wedge (\Delta_v^j - \delta_v^j - \eta_{\alpha\beta}^j \theta_v^\beta) = 0 \mod \varphi, \theta^+,$$

and (4.11) yields

$$(4.30) \quad (\Delta_u^k - \delta_u^k - \eta_u^{\alpha\gamma} \theta_\gamma^k) \wedge \theta_k^\alpha = 0 \quad \text{mod } \varphi, \theta^-.$$

Then using linear independence of θ_α^k , θ_v^α and applying Cartan's Lemma, we obtain

$$\Delta_v^j = \delta_v^j + \eta_{\alpha\beta}^j \theta_v^\beta \quad \text{mod } \{\varphi, \theta^+\}$$

and

$$\Delta_u^k = \delta_u^k + \eta_u^{\alpha\gamma} \theta_\gamma^k \quad \text{mod } \{\varphi, \theta_-\}.$$

Since Δ_v^j is independent of α , we obtain

$$(4.31) \quad \Delta_v^j = \delta_v^j + \eta_\beta^j \theta_v^\beta + \eta_v^\gamma \theta_\gamma^j \quad \text{mod } \varphi,$$

where

$$\eta_\beta^j = \eta_{\alpha\beta}^j, \quad \eta_v^\gamma = \eta_v^{\alpha\gamma}.$$

Hence (4.13) and (4.14) imply

$$\begin{aligned} \Theta_\alpha^j &= \theta_\alpha^j + \eta_\beta^j \varphi_\alpha^\beta, \\ \Theta_u^\alpha &= \theta_u^\alpha + \eta_u^\gamma \varphi_\gamma^\alpha. \end{aligned}$$

Then after applying change of frame of the source manifold given by

$$\begin{aligned} \tilde{Z}_\alpha &= Z_\alpha, \\ \tilde{Z}'_u &= \eta_u^\beta Z_\beta + Z'_u, \\ \tilde{X}_j &= X_j + C_j^\beta Z_\beta, \\ \tilde{Y}_\alpha &= Y_\alpha + A_\alpha^\beta Z_\beta + V_\alpha^v Z'_v + \eta_\alpha^j X_j, \end{aligned}$$

where

$$\begin{aligned} \eta_v^\beta + V_v^\beta &= 0, \\ (A_\alpha^\beta + \overline{A_\beta^\alpha}) - V_\alpha^v V_v^\beta + \eta_\alpha^j \eta_j^\beta &= 0, \\ C_j^\alpha + \eta_j^\alpha &= 0, \end{aligned}$$

we can choose new θ_α^j , θ_u^α and δ_u^j such that

$$(4.32) \quad \Theta_\alpha^j = \theta_\alpha^j,$$

$$(4.33) \quad \Theta_u^\alpha = \theta_u^\alpha,$$

$$(4.34) \quad \Delta_u^j = \delta_u^j \quad \text{mod } \varphi.$$

Lemma 4.1. *Under the assumptions of Theorem 1.3, we have*

$$s = r.$$

Proof. In Lemma 3.3, we showed that

$$\Phi_a^b = 0, \quad a > s \quad \text{or} \quad b > s.$$

Suppose that $s < r$. Choose a tangent vector ξ transversal to T^c such that $\varphi_r^r(\xi) \neq 0$ and $\varphi_\alpha^\beta(\xi) = 0$ for $(\alpha, \beta) \neq (r, r)$. Then $\Phi_a^b(f_*(\xi)) = 0$ for all a, b . Therefore $f_*(\xi) \in T'^c$, which is a contradiction with the assumption of the theorem. \square

4.3. Determination of Θ_α^J , Θ_U^α after a change of frame. Using (4.27) and (4.28) and making a change of frame of the target manifold M' given by

$$\begin{aligned} \tilde{Z}'_U &= Z'_U + \eta_U^\beta Z_\beta, \quad U > q - r \\ \tilde{X}_J &= X_J + C_J^\beta Z_\beta, \quad J > n \\ \tilde{Y}_\alpha &= Y_\alpha + A_\alpha^\beta Z_\beta + \sum_{V > q-r} H_\alpha^V Z'_V + \sum_{J > n} \eta_\alpha^J X_J, \end{aligned}$$

where

$$\begin{aligned} \eta_U^\beta + H_U^\beta &= 0, \quad H_U^\beta := -\overline{H_\beta^U}, \\ (A_\alpha^\beta + \overline{A_\beta^\alpha}) - \sum_{U > q-r} H_\alpha^U H_U^\beta + \sum_{J > n} \eta_\alpha^J \eta_J^\beta &= 0, \\ C_J^\alpha + \eta_J^\alpha &= 0, \end{aligned}$$

and fixing the remaining vectors of the frame, we can obtain new Θ_α^J , Θ_U^α such that

$$(4.35) \quad \Theta_\alpha^J = 0,$$

$$(4.36) \quad \Theta_U^\alpha = 0.$$

Summarizing we obtain

$$(4.37) \quad \Phi_a^b - \varphi_a^b = \Theta_a^J - \theta_a^J = \Theta_a^U - \theta_a^U = 0$$

and hence

$$\begin{aligned} \Psi_a^\beta &= 0 \pmod{\varphi}, \quad a > r, \\ \Delta_v^J &= 0 \pmod{\{\varphi, \theta_\alpha^+\}}, \quad J > n, \\ \Omega_k^J &= 0 \pmod{\{\varphi, \theta_\alpha^+, \theta_\alpha^-\}}, \quad J > n, \\ \Delta_U^k &= 0 \pmod{\{\varphi, \overline{\theta_\alpha^-}\}}, \quad U > q - r, \\ \Omega_U^v &= 0 \pmod{\{\varphi, \overline{\theta_\alpha^+}, \overline{\theta_\alpha^-}\}}, \quad U > q - r. \end{aligned}$$

4.4. Determination of Δ and Ω modulo φ . If $r \geq 2$, i.e. α admits at least two values, then as in [KiZ12], we conclude that the right-hand sides in the previous equations are in fact independent of α . That is,

$$(4.38) \quad \Delta_u^J = \Omega_k^J = \Delta_U^k = \Omega_U^u = 0 \quad \text{mod } \varphi.$$

Suppose now $r = 1$, i.e. $\alpha = 1$. We will analyze the Gauss equations as in [W79]. From above equations and (4.34) we obtain

$$(4.39) \quad \Delta_u^k = \delta_u^k + \eta_u^k \varphi_1^1,$$

$$(4.40) \quad \Delta_u^J = A_u^J \theta_1^k \quad \text{mod } \varphi, \quad J > n,$$

$$(4.41) \quad \Omega_k^J = \tilde{A}_v^J \theta_1^v + B_k^J \theta_1^j \quad \text{mod } \varphi, \quad J > n,$$

$$(4.42) \quad \Delta_U^k = A_U^{kv} \theta_v^1 \quad \text{mod } \varphi, \quad U > q - r,$$

$$(4.43) \quad \Omega_U^u = \tilde{A}_U^{kv} \theta_k^1 + B_U^{uv} \theta_v^1 \quad \text{mod } \varphi, \quad U > q - r.$$

Substituting into (4.16) and (4.18) (with $\eta_\alpha^J = \eta_U^{\alpha\beta} = 0$) we obtain

$$\tilde{A}_v^J = A_v^J, \quad \tilde{A}_U^{kv} = A_U^{kv}, \quad B_k^J = B_l^J, \quad B_U^{uv} = B_U^{vu}.$$

Consider the structure equations $d\pi = \pi \wedge \pi$ obtained by differentiating the following identities from (4.37):

$$\Theta_\alpha^j = \theta_\alpha^j, \quad \Theta_\alpha^u = \theta_\alpha^u.$$

Then we obtain (with $\alpha = 1$):

$$(4.44) \quad \Psi_\alpha^\alpha \wedge \theta_\alpha^j + \theta_\alpha^v \wedge \Delta_v^j + \theta_\alpha^k \wedge \Omega_k^j + \varphi_\alpha^\alpha \wedge \Sigma_\alpha^j = \psi_\alpha^\alpha \wedge \theta_\alpha^j + \theta_\alpha^v \wedge \delta_v^j + \theta_\alpha^k \wedge \omega_k^j + \varphi_\alpha^\alpha \wedge \sigma_\alpha^j,$$

$$(4.45) \quad \Psi_\alpha^\alpha \wedge \theta_\alpha^u + \theta_\alpha^v \wedge \Omega_v^u + \theta_\alpha^k \wedge \Delta_k^u + \varphi_\alpha^\alpha \wedge \Sigma_\alpha^u = \psi_\alpha^\alpha \wedge \theta_\alpha^u + \theta_\alpha^v \wedge \omega_v^u + \theta_\alpha^k \wedge \delta_k^u + \varphi_\alpha^\alpha \wedge \sigma_\alpha^u,$$

which yield using (4.39):

$$(4.46) \quad \left[\hat{\delta}_k^j (\Psi_\alpha^\alpha - \psi_\alpha^\alpha) - (\Omega_k^j - \omega_k^j) \right] \wedge \theta_\alpha^k + \varphi_\alpha^\alpha \wedge (\Sigma_\alpha^j - \sigma_\alpha^j - \eta_v^j \theta_\alpha^v) = 0,$$

$$(4.47) \quad \left[\hat{\delta}_v^u (\Psi_\alpha^\alpha - \psi_\alpha^\alpha) - (\Omega_v^u - \omega_v^u) \right] \wedge \theta_\alpha^v + \varphi_\alpha^\alpha \wedge (\Sigma_\alpha^u - \sigma_\alpha^u - \eta_k^u \theta_\alpha^k) = 0,$$

where

$$\eta_k^u := -\overline{\eta_u^k}$$

and $\hat{\delta}$ denotes the Kronecker delta. Then by Cartan's lemma applied to (4.46), we obtain

$$\hat{\delta}_k^j (\Psi_\alpha^\alpha - \psi_\alpha^\alpha) - (\Omega_k^j - \omega_k^j) = 0 \quad \text{mod } \theta_\alpha^+, \varphi,$$

and by (4.47), we obtain

$$\hat{\delta}_v^u (\Psi_\alpha^\alpha - \psi_\alpha^\alpha) - (\Omega_v^u - \omega_v^u) = 0 \quad \text{mod } \theta_\alpha^-, \varphi,$$

which imply

$$(4.48) \quad \Psi_\alpha^\alpha - \psi_\alpha^\alpha = \Omega_j^j - \omega_j^j = \Omega_u^u - \omega_u^u \quad \text{mod } \theta_\alpha^+, \theta_\alpha^-, \varphi.$$

Using symmetry relation for Ω and the fact that θ_α^+ , θ_α^- are $(1, 0)$, $(0, 1)$ respectively, we obtain

$$\begin{aligned}\Omega_k^j &= \omega_k^j \quad \text{mod } \varphi, & j \neq k, \\ \Omega_v^u &= \omega_v^u \quad \text{mod } \varphi, & u \neq v,\end{aligned}$$

and

$$(4.49) \quad \Psi_\alpha^\alpha - \psi_\alpha^\alpha = \Omega_j^j - \omega_j^j = \Omega_u^u - \omega_u^u \quad \text{mod } \varphi.$$

Now consider the structure equation obtained by differentiating the following identity from (4.37):

$$\Phi_\alpha^\alpha - \varphi_\alpha^\alpha = 0,$$

which yields

$$(4.50) \quad (\Psi_\alpha^\alpha - \psi_\alpha^\alpha - \hat{\Psi}_\alpha^\alpha + \hat{\psi}_\alpha^\alpha) \wedge \varphi_\alpha^\alpha = 0,$$

or equivalently

$$(\Psi_\alpha^\alpha - \psi_\alpha^\alpha + \Psi_{\bar{\alpha}}^{\bar{\alpha}} - \psi_{\bar{\alpha}}^{\bar{\alpha}}) \wedge \varphi_\alpha^\alpha = 0.$$

Let

$$\Psi_\alpha^\alpha - \psi_\alpha^\alpha = \hat{\Psi}_\alpha^\alpha - \hat{\psi}_\alpha^\alpha + g\varphi_\alpha^\alpha$$

for some pure imaginary function g . Applying a real vector change (see Defintion 2.1) of the source manifold defined by

$$\tilde{Y}_\alpha = Y_\alpha + \frac{g}{2}Z_\alpha,$$

we may assume that

$$(4.51) \quad \Psi_\alpha^\alpha - \psi_\alpha^\alpha = \hat{\Psi}_\alpha^\alpha - \hat{\psi}_\alpha^\alpha.$$

By (4.46), (4.47), we obtain

$$(4.52) \quad \Sigma_\alpha^j - \sigma_\alpha^j = g_k^j \theta_\alpha^k + \eta_v^j \theta_\alpha^v \quad \text{mod } \varphi,$$

$$(4.53) \quad \Sigma_\alpha^u - \sigma_\alpha^u = g_v^u \theta_\alpha^v + \eta_k^u \theta_\alpha^k \quad \text{mod } \varphi,$$

for suitable functions g_k^j , g_v^u . Then, using the structure identities $d\pi = \pi \wedge \pi$ obtained from differentiating (4.51), we obtain

$$\theta_\alpha^v \wedge (\Sigma_u^\alpha - \sigma_u^\alpha) + \theta_\alpha^k \wedge (\Sigma_k^\alpha - \sigma_k^\alpha) = (\Sigma_\alpha^v - \sigma_\alpha^v) \wedge \theta_v^\alpha + (\Sigma_\alpha^k - \sigma_\alpha^k) \wedge \theta_k^\alpha \quad \text{mod } \varphi,$$

which implies

$$g_k^j = \overline{g_j^k} \quad g_u^v = \overline{g_v^u}.$$

Then substituting (4.52), (4.53) into (4.46) and (4.47) imply

$$(4.54) \quad \hat{\delta}_k^j (\Psi_\alpha^\alpha - \psi_\alpha^\alpha) - (\Omega_k^j - \omega_k^j) + g_j^k \varphi_\alpha^\alpha = 0,$$

$$(4.55) \quad \hat{\delta}_v^u (\Psi_\alpha^\alpha - \psi_\alpha^\alpha) - (\Omega_v^u - \omega_v^u) + g_v^u \varphi_\alpha^\alpha = 0.$$

Now differentiate (4.54) and use the structure equations $d\pi = \pi \wedge \pi$ together with (4.39), (4.49) to obtain

$$\begin{aligned} & \sum_{V>q-r} \Delta_k^V \wedge \Delta_V^j + \sum_{K>n} \Omega_k^K \wedge \Omega_K^j \\ &= \hat{\delta}_k^j (\theta_\alpha^v \wedge (\Sigma_v^\alpha - \sigma_v^\alpha) + \theta_\alpha^l \wedge (\Sigma_l^\alpha - \sigma_l^\alpha)) + \hat{\delta}_l^j (\theta_\alpha^l \wedge (\Sigma_k^\alpha - \sigma_k^\alpha) - \theta_k^\alpha \wedge (\Sigma_\alpha^l - \sigma_\alpha^l)) \\ &+ g_k^j (\theta_\alpha^v \wedge \theta_v^\alpha + \theta_\alpha^l \wedge \theta_l^\alpha) \quad \text{mod } \varphi. \end{aligned}$$

Substituting the identities following (4.39) as well as (4.52), (4.53), we now obtain

$$(4.56) \quad B_{kl}^K B_K^{jm} = g_l^m \hat{\delta}_k^j + g_l^j \hat{\delta}_k^m + g_k^m \hat{\delta}_l^j + g_k^j \hat{\delta}_l^m,$$

$$(4.57) \quad A_{uk}^K B_K^{jm} = \eta_u^m \hat{\delta}_k^j + \eta_u^j \hat{\delta}_k^m,$$

$$(4.58) \quad A_{ku}^V A_V^{jv} + A_{uk}^K A_K^{vj} = g_u^v \hat{\delta}_k^j + g_k^j \hat{\delta}_u^v.$$

Now Lemma 5.3 from [EHZ05] implies that $B_k^K = 0$ provided $n' < 2n$, which is part of our assumptions (recall that $n = p - r$). Therefore putting all indices to be j in the right-hand side of (4.56) we obtain $g_j^j = 0$ and hence $g_j^k = 0$ since it is hermitian. Then (4.57) for $m = k = j$ yields

$$\eta_u^j = 0.$$

Then (4.41) reads

$$\Omega_k^J = A_{uk}^J \theta_\alpha^u \quad \text{mod } \varphi, \quad J > n.$$

Then by differentiation and the structure identities $d\pi = \pi \wedge \pi$ we obtain

$$\Delta_k^V \wedge \Delta_V^J + \Omega_k^L \wedge \Omega_L^J + \theta_k^\alpha \wedge \Sigma_\alpha^J = A_{uk}^J \theta_\alpha^l \wedge \delta_l^u \quad \text{mod } \theta^-, \varphi.$$

By substituting (4.39) and (4.40), we obtain

$$A_{ul}^J \delta_k^u \wedge \theta_1^l = A_{uk}^J \theta_1^l \wedge \delta_l^u.$$

Hence

$$A_{uk}^J = 0$$

and (4.58) becomes

$$\sum_{V>q-r} A_{lu}^V A_V^{jw} = g_u^w \hat{\delta}_l^j.$$

Therefore for each fixed u , the vectors $A^{ju} := (A_V^{ju}) \in \mathbb{C}^{q'-r'-(q-r)}$, $j = 1, \dots, n$, are orthogonal to each other. If

$$g_u^u \neq 0,$$

then

$$A_{ju}^V A_V^{ju} = g_u^u \neq 0, \quad j = 1, \dots, n,$$

which leads to a contradiction, since we assumed $q' - r' < n$. Consequently $A_{ju}^V = 0$, $g_u^u = 0$ and hence $g_u^v = 0$, and (4.42), (4.43) yield

$$\Delta_U^j = 0 \quad \text{mod } \varphi, \quad U > q - r$$

and

$$\Omega_U^v = B_U^{vw} \theta_w^\alpha \quad \text{mod } \varphi, \quad U > q - r.$$

By differentiating the last equation and following the same argument as before for Ω_k^J , we obtain

$$B_U^{vw} = 0.$$

Summing up, we obtain

Lemma 4.2.

$$\begin{aligned}\Delta_u^j &= \delta_u^j, \\ \Psi_a^\beta &= 0 \pmod{\varphi}, \quad a > r, \\ \Delta_v^J &= \Omega_k^J = 0 \pmod{\varphi} \quad J > n, \\ \Delta_U^k &= \Omega_U^v = 0 \pmod{\varphi}, \quad U > q - r.\end{aligned}$$

5. DETERMINATION OF THE SECOND FUNDAMENTAL FORMS

Next, we shall determine all second fundamental forms

$$\begin{aligned}\Psi_a^\beta, \quad a > r, \\ \Delta_v^J, \Omega_k^J, \Sigma_\beta^J \quad J > n, \\ \Delta_U^k, \Omega_U^v, \Sigma_U^\beta \quad U > q - r.\end{aligned}$$

5.1. Determination of Ψ_a^β for $a > r$. In view of Lemma 4.2 we can write

$$(5.1) \quad \Psi_a^\beta = h_a^{\beta\gamma} \varphi_\gamma^\delta, \quad a > r.$$

Since

$$\Phi_a^b = \Theta_a^J = \Theta_a^U = 0, \quad a > r,$$

differentiation of (5.1) and using the structure equations $d\pi = \pi \wedge \pi$ yields

$$h_a^{\beta\gamma} (\theta_\gamma^u \wedge \theta_u^\delta + \theta_\gamma^k \wedge \theta_k^\delta) = 0 \pmod{\varphi},$$

which implies

$$h_a^{\beta\gamma} = 0,$$

and hence

$$(5.2) \quad \Psi_a^\beta = 0, \quad a > r.$$

5.2. Determination of Δ_u^J, Ω_k^J for $J > n$. In view of Lemma 4.2 we write

$$\Delta_u^J = h_u^{J\alpha} \varphi_\alpha^\beta, \quad J > n.$$

By differentiation and using structure identities as before, we obtain

$$\theta_u^\beta \wedge \Sigma_\beta^J = h_u^{J\alpha} (\theta_\alpha^v \wedge \theta_v^\beta + \theta_\alpha^k \wedge \theta_k^\beta) \pmod{\varphi}.$$

Since the left hand side contains no $\theta_\alpha^k \wedge \theta_k^\beta$ terms, we obtain

$$h_u^{J\alpha} = 0,$$

i.e.,

$$\Delta_u^J = 0$$

and

$$\Sigma_\beta^J = 0 \quad \text{mod } \theta_u^\beta, \varphi.$$

Similar argument implies

$$\Omega_k^J = 0$$

and

$$\Sigma_\beta^J = 0 \quad \text{mod } \theta_k^\beta, \varphi.$$

Consequently, we obtain

$$\Sigma_\beta^J = 0 \quad \text{mod } \varphi.$$

5.3. Determination of Δ_U^j , Ω_U^v for $U > q - r$. Let

$$\Delta_U^j = h_U^j \alpha_\beta \varphi_\alpha^\beta.$$

By differentiation and structure identities as before, we obtain

$$\Sigma_U^\beta \wedge \theta_\beta^j = h_U^j \alpha_\beta \left(\theta_\alpha^v \wedge \theta_v^\beta + \theta_\alpha^k \wedge \theta_k^\beta \right) \quad \text{mod } \varphi.$$

Since the left hand side contains no $\theta_\alpha^v \wedge \theta_v^\beta$ terms, we obtain

$$h_U^j \alpha_\beta = 0,$$

i.e.,

$$\Delta_U^j = 0$$

and

$$\Sigma_U^\beta = 0 \quad \text{mod } \theta_\beta^j, \varphi.$$

Similar argument implies

$$\Omega_U^v = 0$$

and

$$\Sigma_U^\beta = 0 \quad \text{mod } \theta_\beta^v, \varphi.$$

Consequently, we obtain

$$\Sigma_U^\beta = 0 \quad \text{mod } \varphi.$$

5.4. Determination of Σ_β^J and Σ_U^β for $U > q - r$ and $J > n$. Now let

$$\Sigma_\gamma^J = h_\gamma^J \alpha_\beta \varphi_\alpha^\beta.$$

By differentiation, we obtain

$$0 = h_\gamma^J \alpha_\beta \left(\theta_\alpha^v \wedge \theta_v^\beta + \theta_\alpha^k \wedge \theta_k^\beta \right) \quad \text{mod } \varphi,$$

which implies

$$h_\gamma^J \alpha_\beta = 0,$$

i.e.,

$$\Sigma_\gamma^J = 0.$$

Similar argument implies

$$\Sigma_U^\beta = 0.$$

We summarize the obtained alignment of the connection forms:

Proposition 5.1. *For any local CR embedding f from $S_{p,q,r}$ into $S_{p',q',r'}$ satisfying the assumptions of either Theorem 1.3, there is a choice of sections of the frame bundles $\mathcal{B}_{p,q} \rightarrow S_{p,q}$ and $\mathcal{B}_{p',q'} \rightarrow S_{p',q'}$ such that*

$$(5.3) \quad \Phi_a^b = \varphi_a^b, \quad \Theta_U^a = \theta_U^a, \quad \Theta_a^J = \theta_a^J, \quad \Delta_u^j = \delta_u^j,$$

$$(5.4) \quad \Psi_a^\beta = \Delta_u^K = \Delta_U^j = \Omega_j^K = \Omega_U^v = \Sigma_\alpha^K = \Sigma_U^\beta = 0, \quad a > r, \quad U > q - r, \quad K > n.$$

6. SPLITTING OF THE IMAGE WITH SUITABLE DIMENSIONS

We shall write $Gr(V, s)$ for the Grassmanian of all s -dimensional subspaces of V .

Proposition 6.1. *Under the assumptions of Theorem 1.3, there exist vector subspaces $V_0, V_1, V_2 \subset \mathbb{C}^{p'+q'}$ of dimension*

$$\dim V_0 = p + q, \quad \dim V_1 = r' - r, \quad \dim V_2 = n' + q' - r' - (n + q - r)$$

that form a direct sum, such that the basic form $\langle \cdot, \cdot \rangle$ is null when restricted to V_1 , nondegenerate of signature (p, q) when restricted to V_0 , and nondegenerate of signature $(n' - n, q' - r' - q + r)$ when restricted to V_2 , and such that whenever $x \in S_{p,q,r}$ and $f(x)$ is defined, we have

$$(6.1) \quad f(x) = W_0 \oplus V_1 \oplus W_2 \in Gr(V_0, q) \oplus V_1 \oplus Gr(V_2, (q' - r') - (q - r)),$$

such that the basic form restricted to W_0 has rank r .

Proof. Denote by $M \subset S_{p,q,r}$ the open subset where f is defined. Let Z, Z', X, Y be collections of constant vector fields valued in $\mathbb{C}^{p'+q'}$ as in Section 2.1, forming a $S_{p',q',r'}$ -frame adapted to $f(M)$ at a fixed reference point in $f(M)$. Let

$$(6.2) \quad \tilde{Z}_a = \lambda_a^b Z_b + \mu_a^V Z'_V + \eta_a^K X_K + \zeta_a^b Y_b,$$

$$(6.3) \quad \tilde{Z}'_U = \lambda_U^b Z_b + \mu_U^V Z'_V + \eta_U^K X_K + \zeta_U^b Y_b,$$

$$(6.4) \quad \tilde{X}_J = \lambda_J^b Z_b + \mu_J^V Z'_V + \eta_J^K X_K + \zeta_J^b Y_b,$$

$$(6.5) \quad \tilde{Y}_a = \hat{\lambda}_a^b Z_b + \hat{\mu}_a^V Z'_V + \hat{\eta}_a^K X_K + \hat{\zeta}_a^b Y_b$$

be an adapted $S_{p',q',r'}$ -frame along $f(M)$. Set

$$(6.6) \quad A := \begin{pmatrix} \lambda_a^b & \mu_a^V & \eta_a^K & \zeta_a^b \\ \lambda_U^b & \mu_U^V & \eta_U^K & \zeta_U^b \\ \lambda_J^b & \mu_J^V & \eta_J^K & \zeta_J^b \\ \hat{\lambda}_a^b & \hat{\mu}_a^V & \hat{\eta}_a^K & \hat{\zeta}_a^b \end{pmatrix},$$

so that (6.2) - (6.5) take the form

$$(6.7) \quad \begin{pmatrix} \tilde{Z}_a \\ \tilde{Z}'_U \\ \tilde{X}_J \\ \tilde{Y}_a \end{pmatrix} = A \begin{pmatrix} Z_b \\ Z'_V \\ X_K \\ Y_b \end{pmatrix}.$$

Since Z, Z', X, Y form an adapted frame at a reference point of $f(M)$, we may assume that

$$(6.8) \quad A = I_{p'+q'}$$

at the reference point. Since Z, Z', X, Y are constant vector fields, i.e., $dZ = dZ' = dX = dY = 0$, differentiating (6.7) and using (2.7) for $\tilde{Z}, \tilde{Z}', \tilde{X}, \tilde{Y}$ we obtain

$$(6.9) \quad dA = \Pi A,$$

where Π is the connection matrix of $S_{p',q',r'}$, i.e. we have

$$(6.10) \quad dA = \begin{pmatrix} \Psi_a^b & \Theta_a^V & \Theta_a^K & \Phi_a^b \\ \Sigma_U^b & \Omega_U^V & \Delta_U^K & \Theta_U^b \\ \Sigma_J^b & \Omega_J^V & \Omega_J^K & \Theta_J^b \\ \Xi_a^b & \Sigma_a^V & \Sigma_a^K & \hat{\Psi}_a^b \end{pmatrix} A.$$

Next, it follows from Proposition 5.1 that

$$(6.11) \quad d\tilde{Z}_a = \sum_{b>r} \Psi_a^b \tilde{Z}_b, \quad a > r,$$

in particular, the span of \tilde{Z}_a , $a > r$, is independent of the point in $f(M)$. Therefore together with (6.2) and (6.8), we conclude

$$(6.12) \quad \mu_a^V = \eta_a^K = \zeta_a^b = 0, \quad a > r.$$

Then using again Proposition 5.1,

$$(6.13) \quad d\tilde{Z}_U = \sum_{b>r} \Sigma_U^b Z_b + \sum_{V>q-r} \Omega_U^V Z_V + \sum_{K>n} \Delta_U^K X_K, \quad U > q - r,$$

$$(6.14) \quad d\tilde{X}_J = \sum_{b>r} \Sigma_J^b Z_b + \sum_{V>q-r} \Delta_J^V Z_V + \sum_{K>n} \Omega_J^K X_K, \quad J > n,$$

which together with (6.11) imply that the span of $\tilde{Z}_a, \tilde{Z}'_U, \tilde{X}_J$ is the same as the span of Z_a, Z'_U, X_J where $a > r, U > q - r, J > n$. Let

$$V_1 := \text{span} \{Z_a : a > r\} = \text{span} \{\tilde{Z}_a : a > r\}.$$

Consider hermitian form $\langle \cdot, \cdot \rangle$ as in (2.1). By definition of adapted frame, $\langle \cdot, \cdot \rangle$ restricted to V_1 is null. Choose V_2 transversal to V_1 such that

$$V_2 = \text{span} \{Z'_U, X_J : U > q - r, J > n\}.$$

Then V_1 is the kernel of $\langle \cdot, \cdot \rangle|_{V_1 \oplus V_2}$ and $\langle \cdot, \cdot \rangle$ restricted to V_2 is nondegenerate with $q' - r' - (q - r)$ negative and $n' - n$ positive eigenvalues. Furthermore, (6.10) implies

$$(6.15) \quad \begin{pmatrix} d\mu_a^V \\ d\mu_U^V \\ d\mu_J^V \\ d\hat{\mu}_a^V \end{pmatrix} = \begin{pmatrix} \Psi_a^b & \Theta_a^W & \Theta_a^L & \Phi_a^b \\ \Sigma_U^b & \Omega_U^W & \Delta_U^L & \Theta_U^b \\ \Sigma_J^b & \Omega_J^W & \Omega_J^L & \Theta_J^b \\ \Xi_a^b & \Sigma_a^W & \Sigma_a^L & \hat{\Psi}_a^b \end{pmatrix} \begin{pmatrix} \mu_b^V \\ \mu_W^V \\ \mu_L^V \\ \hat{\mu}_b^V \end{pmatrix},$$

$$(6.16) \quad \begin{pmatrix} d\eta_a^K \\ d\eta_U^K \\ d\eta_J^K \\ d\hat{\eta}_a^K \end{pmatrix} = \begin{pmatrix} \Psi_a^b & \Theta_a^W & \Theta_a^L & \Phi_a^b \\ \Sigma_U^b & \Omega_U^W & \Delta_U^L & \Theta_U^b \\ \Sigma_J^b & \Omega_J^W & \Omega_J^L & \Theta_J^b \\ \Xi_a^b & \Sigma_a^W & \Sigma_a^L & \hat{\Psi}_a^b \end{pmatrix} \begin{pmatrix} \eta_b^K \\ \eta_W^K \\ \eta_L^K \\ \hat{\eta}_b^K \end{pmatrix}$$

and

$$(6.17) \quad \begin{pmatrix} d\zeta_a^c \\ d\zeta_U^c \\ d\zeta_J^c \\ d\hat{\zeta}_a^c \end{pmatrix} = \begin{pmatrix} \Psi_a^b & \Theta_a^W & \Theta_a^L & \Phi_a^b \\ \Sigma_U^b & \Omega_U^W & \Delta_U^L & \Theta_U^b \\ \Sigma_J^b & \Omega_J^W & \Omega_J^L & \Theta_J^b \\ \Xi_a^b & \Sigma_a^W & \Sigma_a^L & \hat{\Psi}_a^b \end{pmatrix} \begin{pmatrix} \zeta_b^c \\ \zeta_W^c \\ \zeta_L^c \\ \hat{\zeta}_b^c \end{pmatrix}.$$

Now restricting to $a = \alpha \leq r$, $U = u \leq q - r$ and $J = j \leq n$, $V > q - r$, $K > n$, $c > r$, and using (6.12) and Proposition 5.1 taken into account, we obtain

$$(6.18) \quad \begin{pmatrix} d\mu_\alpha^V \\ d\mu_u^V \\ d\mu_j^V \\ d\hat{\mu}_\alpha^V \end{pmatrix} = \begin{pmatrix} \Psi_\alpha^\beta & \Theta_\alpha^w & \Theta_\alpha^l & \Phi_\alpha^\beta \\ \Sigma_u^\beta & \Omega_u^w & \Delta_u^l & \Theta_u^\beta \\ \Sigma_j^\beta & \Omega_j^w & \Omega_j^l & \Theta_j^\beta \\ \Xi_\alpha^\beta & \Sigma_\alpha^w & \Sigma_\alpha^l & \hat{\Psi}_\alpha^\beta \end{pmatrix} \begin{pmatrix} \mu_\beta^V \\ \mu_w^V \\ \mu_k^V \\ \hat{\mu}_\beta^V \end{pmatrix},$$

$$(6.19) \quad \begin{pmatrix} d\eta_\alpha^K \\ d\eta_u^K \\ d\eta_j^K \\ d\hat{\eta}_\alpha^K \end{pmatrix} = \begin{pmatrix} \Psi_\alpha^\beta & \Theta_\alpha^w & \Theta_\alpha^l & \Phi_\alpha^\beta \\ \Sigma_u^\beta & \Omega_u^w & \Delta_u^l & \Theta_u^\beta \\ \Sigma_j^\beta & \Omega_j^w & \Omega_j^l & \Theta_j^\beta \\ \Xi_\alpha^\beta & \Sigma_\alpha^w & \Sigma_\alpha^l & \hat{\Psi}_\alpha^\beta \end{pmatrix} \begin{pmatrix} \eta_\beta^K \\ \eta_w^K \\ \eta_l^K \\ \hat{\eta}_\beta^K \end{pmatrix}$$

and

$$(6.20) \quad \begin{pmatrix} d\zeta_\alpha^c \\ d\zeta_u^c \\ d\zeta_j^c \\ d\hat{\zeta}_\alpha^c \end{pmatrix} = \begin{pmatrix} \Psi_\alpha^\beta & \Theta_\alpha^w & \Theta_\alpha^l & \Phi_\alpha^\beta \\ \Sigma_u^\beta & \Omega_u^w & \Delta_u^l & \Theta_u^\beta \\ \Sigma_j^\beta & \Omega_j^w & \Omega_j^l & \Theta_j^\beta \\ \Xi_\alpha^\beta & \Sigma_\alpha^w & \Sigma_\alpha^l & \hat{\Psi}_\alpha^\beta \end{pmatrix} \begin{pmatrix} \zeta_\beta^c \\ \zeta_w^c \\ \zeta_l^c \\ \hat{\zeta}_\beta^c \end{pmatrix}.$$

Thus each of the vector valued functions $\mu^V = (\mu_\alpha^V, \mu_u^V, \mu_j^V, \hat{\mu}_\beta^V)$, $\eta^K := (\eta_\alpha^K, \eta_u^K, \eta_j^K, \hat{\eta}_\beta^K)$ and $\zeta^c := (\zeta_\alpha^c, \zeta_u^c, \zeta_j^c, \hat{\zeta}_\beta^c)$ for fixed $V > q - r$, $K > n$ and $c > r$ satisfies a complete system of first order linear differential equations. Then by the initial condition (6.8) and the uniqueness of solutions, we conclude, in particular, that

$$(6.21) \quad \mu^V = \eta^K = \zeta^c = 0, \quad V > q - r, \quad K > n, \quad c > r.$$

Hence (6.2), (6.3) imply

$$(6.22) \quad \tilde{Z}_\alpha = \lambda_\alpha^b Z_b + \mu_\alpha^v Z'_v + \eta_\alpha^k X_k + \zeta_\alpha^\beta Y_\beta,$$

$$(6.23) \quad \tilde{Z}'_u = \lambda_u^b Z_b + \mu_u^v Z'_v + \eta_u^k X_k + \zeta_u^\beta Y_\beta.$$

Now setting

$$(6.24) \quad \widehat{Z}_\alpha := \widetilde{Z}_\alpha - \sum_{b>r} \lambda_\alpha^b Z_b,$$

$$(6.25) \quad \widehat{Z}'_u := \widetilde{Z}'_u - \sum_{b>r} \lambda_u^b Z_b,$$

$$(6.26) \quad \widehat{Z}'_U := \widetilde{Z}'_U - \sum_{b>r} \lambda_U^b Z_b,$$

we still have

$$(6.27) \quad \text{span} \{ \widehat{Z}_\alpha, \widetilde{Z}_{r+1}, \dots, \widetilde{Z}_{r'}, \widehat{Z}'_U \} = \text{span} \{ \widetilde{Z}_a, \widetilde{Z}'_U \},$$

whereas (6.22), (6.23) and (6.3) become

$$(6.28) \quad \widehat{Z}_\alpha = \lambda_\alpha^\beta Z_\beta + \mu_\alpha^v Z'_v + \eta_\alpha^k X_k + \zeta_\alpha^\beta Y_\beta,$$

$$(6.29) \quad \widehat{Z}'_u = \lambda_u^\beta Z_\beta + \mu_u^v Z'_v + \eta_u^k X_k + \zeta_u^\beta Y_\beta$$

$$(6.30) \quad \widehat{Z}'_U = \lambda_U^\beta Z_\beta + \mu_U^V Z'_V + \eta_U^K X_K + \zeta_U^b Y_b, \quad U > q - r,$$

implying

$$\text{span} \{ \widehat{Z}_\alpha, \widehat{Z}'_u \} \subset \text{span} \{ Z_\alpha, Z'_u, X_k, Y_\beta \} =: V_0,$$

and since \widetilde{Z}'_U is in the span of Z_a, Z'_V, X_J with $a > r$, $V > q - r$, $J > n$,

$$\widehat{Z}'_U = \sum_{V>q-r} \mu_U^V Z'_V + \sum_{K>n} \eta_U^K X_K, \quad U > q - r,$$

implying

$$\text{span} \{ \widehat{Z}'_U \} \subset \text{span} \{ Z'_U, X_J : U > q - r, J > n \} = V_2.$$

Then together with (6.11) we conclude that for $x \in M$,

$$\begin{aligned} f(x) &= \text{span} \{ \widetilde{Z}_a, \widetilde{Z}'_U \} = \text{span} \{ \widehat{Z}_\alpha, \widehat{Z}'_u \} \oplus \text{span} \{ \widetilde{Z}_{r+1}, \dots, \widetilde{Z}_{r'} \} \oplus \text{span} \{ \widehat{Z}'_{q-r+1}, \dots, \widehat{Z}'_{q'-r'} \} \\ &= \text{span} \{ \widehat{Z}_\alpha, \widehat{Z}'_u \} \oplus \text{span} \{ Z_{r+1}, \dots, Z_{r'} \} \oplus \text{span} \{ \widehat{Z}'_{q-r+1}, \dots, \widehat{Z}'_{q'-r'} \} \\ &\in Gr(V_0, q) \oplus V_1 \oplus Gr(V_2, (q' - r') - (q - r)). \end{aligned}$$

□

7. CLASSIFICATION OF CR MAPS BETWEEN BOUNDARY COMPONENTS

Proof of Theorem 1.3. Let V_0 be given by Proposition 6.1. After a linear change of coordinates in $\mathbb{C}^{p'+q'}$ preserving the basic form (that corresponds to an automorphism of $D_{p',q'}$), we may assume that $V_0 = \mathbb{C}^{p+q} \times \{0\}$ and hence the $Gr(V_0, q)$ -component of f in (6.1) defines a local CR diffeomorphism of $S_{p,q,r}$. Then by a theorem of Kaup-Zaitsev [KaZ06, Theorem 4.5], the $Gr(V_0, q)$ -component of f is a restriction of a global CR-automorphism of $S_{p,q,r}$. Furthermore, by [KaZ00, Theorem 8.5], the $Gr(V_0, q)$ -component of f extends to a biholomorphic automorphism of the bounded symmetric domain $D_{p,q}$. Hence, composing f with a suitable automorphism of $D_{p',q'}$ we can put f in the form (1.6). Since $f(x) \in S_{p',q',r'}$, it follows from the description of $S_{p',q',r'}$ that in the notation of (1.6) we

must have (1.7). Vice versa, any f of the form (1.6) with h satisfying (1.7) defines a CR map between pieces of $S_{p,q,r}$ and $S_{p',q',r'}$ satisfying the assumptions of Theorem 1.3. The proof is complete. \square

Proof of Theorem 1.1. Let f be as in the corollary. Consider the restriction \tilde{f} of f to the hypersurface boundary component $S_{p,q,1}$. Then \tilde{f} restricts to a CR map between open pieces of $S_{p,q,1}$ and $S_{p',q',r'}$ for some $1 \leq r' \leq q'$. Since $S_{p,q,1}$ is a real hypersurface, the transversality assumption $df(\xi) \in T' \setminus T'^c$ for $\xi \in T \setminus T^c$ of Theorem 1.3 is satisfied. Indeed, otherwise we would have $df(T) \subset T'^c$ and the Levi form identity (2.41) would imply that $df(T)$ is contained in the Levi null-space of $S_{p',q',r'}$, which, in view of positivity, coincides with the kernel. The latter is an integrable distribution whose orbits are complex submanifolds of $S_{p',q',r'}$. Then f would send any curve in $S_{p,q,1}$ into one of these complex submanifolds (see [BER99] for details). Hence it would follow that f sends an open piece of $S_{p,q,1}$ into a complex submanifold of $S_{p',q',r'}$, which would contradict the assumptions of corollary.

Next, since $r' \geq r = 1$, the assumptions (1.4) imply (1.5). Now by Theorem 1.3, we can assume that f is of the form (1.6). Furthermore, the assumption $f(U \cap D_{p,q}) \not\subset \partial D_{p',q'}$ implies that the block $I_{r'-r}$ in (1.6) must be trivial, and hence f is of the desired form. \square

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